More applications of change of variables

Example 1. **(Mass)** Consider a rigid body occupying $\Omega \subseteq \mathbb{R}^3$ with density $\rho(x, y, z)$. Then its mass is given by

$$\int_\Omega \rho(x, y, z)\, \mathrm{d}(x, y, z). \tag{1}$$

Example 2. **(Center of mass)** Consider a rigid body occupying $\Omega \subseteq \mathbb{R}^3$ with density $\rho(x, y, z)$. Its center of mass is the unique point $\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \in \mathbb{R}^3$ such that

$$\int_\Omega \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} \rho(x, y, z)\, \mathrm{d}(x, y, z) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{2}$$

We see that

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \left(\int_\Omega \rho(x, y, z)\, \mathrm{d}(x, y, z) \right)^{-1} \int_\Omega \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rho(x, y, z)\, \mathrm{d}(x, y, z). \tag{3}$$

Exercise 1. Prove that if Ω is convex, then for any density $\rho(x, y, z)$ the center of mass is in Ω.

Example 3. **(Angular momentum)** Consider a rigid body occupying $\Omega \subseteq \mathbb{R}^3$ with density $\rho(x, y, z)$. Then its angular momenta around x, y, z axes are

$$I_x = \int_\Omega (y^2 + z^2)\, \rho(x, y, z)\, \mathrm{d}(x, y, z); \tag{4}$$

$$I_y = \int_\Omega (x^2 + z^2)\, \rho(x, y, z)\, \mathrm{d}(x, y, z); \tag{5}$$

$$I_z = \int_\Omega (x^2 + y^2)\, \rho(x, y, z)\, \mathrm{d}(x, y, z). \tag{6}$$

In general, its angular momentum around a straight line L is given by

$$I_L = \int_\Omega [\mathrm{dist}((x, y, z), L)]^2\, \rho(x, y, z)\, \mathrm{d}(x, y, z). \tag{7}$$

Now let L be a straight line parallel to z-axis and the distance between them is h. Find the relation between I_L and I_z.

Let L intersect x-y plane at (x_0, y_0). Then $x_0^2 + y_0^2 = h^2$ and

$$[\mathrm{dist}((x, y, z), L)]^2 = (x - x_0^2) + (y - y_0)^2. \tag{8}$$

Thus we have

$$\begin{aligned} I_L &= \int_\Omega [(x - x_0^2) + (y - y_0)^2]\, \rho(x, y, z)\, \mathrm{d}(x, y, z) \\ &= \int_\Omega (x^2 + y^2)\, \rho(x, y, z)\, \mathrm{d}(x, y, z) - 2\, x_0 \int_\Omega x\, \rho(x, y, z)\, \mathrm{d}(x, y, z) \\ &\quad -2\, y_0 \int_\Omega y\, \rho(x, y, z)\, \mathrm{d}(x, y, z) + (x_0^2 + y_0^2) \int_\Omega \rho(x, y, z)\, \mathrm{d}(x, y, z). \end{aligned} \tag{9}$$

Thus if we choose z-axis such that it passes the center of mass, we would have

$$I_L = I_z + h^2 M \tag{10}$$

where h is the distance between L and the z-axis, and M is the mass.

Example 4. (Gravity) Let Ω be a body in space with density $\rho(x, y, z)$. Then the gravity force it generates on a unit mass at (x_0, y_0, z_0) is

$$\boldsymbol{F} = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = \int_\Omega \frac{K\,\rho(x, y, z)}{r^3} \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} \mathrm{d}(x, y, z) \tag{11}$$

where $r = [(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{1/2}$.

Now assume Ω is a ball

$$\Omega := \{(x, y, z) \in \mathbb{R}^N | \, 0 \leqslant x^2 + y^2 + z^2 \leqslant R^2\} \tag{12}$$

with uniform density $\rho(x, y, z) = \rho$. We determine the gravity force on a unit mass at (x_0, y_0, z_0) inside the shell. Wlog, we can assume $x_0 = y_0 = 0$, and denote $z_0 = l > 0$.

Now we have

$$F_x = K\,\rho \int_\Omega \frac{x}{r^3} \mathrm{d}(x, y, z) = K\,\rho \left[\int_{\Omega \cap \{x > 0\}} \frac{x}{r^3} \mathrm{d}(x, y, z) + \int_{\Omega \cap \{x < 0\}} \frac{x}{r^3} \mathrm{d}(x, y, z) \right]. \tag{13}$$

Now apply change of variable $T(u, v, w) = (-u, v, w)$ that is $x = -u, y = v, z = w$, we see that

$$\int_{\Omega \cap \{x < 0\}} \frac{x}{r^3} \mathrm{d}(x, y, z) = -\int_{\Omega \cap \{x > 0\}} \frac{u}{r^3} \mathrm{d}(u, v, z) \tag{14}$$

so $F_x = 0$.

Similarly $F_y = 0$.

We have

$$F_z = K\,\rho \int_\Omega \frac{z - l}{[x^2 + y^2 + (z - l)^2]^{3/2}} \mathrm{d}(x, y, z). \tag{15}$$

Applying cylindrical coordinates we have

$$\begin{aligned} F_z &= K\,\rho \int_{-R}^{R} \left[\int_{0 \leqslant x^2 + y^2 \leqslant R^2 - z^2} \frac{z - l}{[x^2 + y^2 + (z - l)^2]^{3/2}} \mathrm{d}(x, y) \right] \mathrm{d}z \\ &= 2\,\pi\,K\,\rho \int_{-R}^{R} \left[\int_0^{\sqrt{R^2 - z^2}} \frac{z - l}{(r^2 + (z - l)^2)^{3/2}} r\,\mathrm{d}r \right] \mathrm{d}z \\ &= 2\,\pi\,K\,\rho \int_{-R}^{R} \left[\frac{z - l}{|z - l|} - \frac{z - l}{(R^2 + l^2 - 2\,l\,z)^{1/2}} \right] \mathrm{d}z \\ &:= 2\,\pi\,K\,\rho\,(I_1 - I_2). \end{aligned} \tag{16}$$

Now for I_1 it is easy to see

$$I_1 = \begin{cases} -2\,R & R \leqslant l \\ -2\,l & R \geqslant l \end{cases}. \tag{17}$$

On the other hand, through integration by parts we hae

$$
\begin{aligned}
I_2 &= \int_{-R}^{R} \frac{z-l}{(R^2+l^2-2\,l\,z)^{1/2}}\,\mathrm{d}z \\
&= \left[-\frac{z-l}{l}\,(R^2+l^2-2\,l\,z)^{1/2}\right]_{-R}^{R} + \frac{1}{l}\int_{-R}^{R} (R^2+l^2-2\,l\,z)^{1/2}\,\mathrm{d}z \\
&= \frac{1}{l}\,[-(R-l)\,|R-l|-(R+l)^2] - \frac{1}{3\,l^2}\,[|R-l|^3-(R+l)^3] \\
&= \begin{cases} \dfrac{2\,R^3}{3\,l^2} - 2\,R & R \leqslant l \\ -\dfrac{4}{3}\,l & R \geqslant l \end{cases}.
\end{aligned} \tag{18}
$$

Summarizing, we have

$$
F_z = \begin{cases} -\dfrac{K\,M}{l^2} & R \leqslant l \\ -\dfrac{K\,M_l}{l^2} & R \geqslant l \end{cases}. \tag{19}
$$

Exercise 2. Calculate the gravity force of a shell:

$$
\Omega := \{(x,y,z)|\,0 < R_1^2 \leqslant x^2+y^2+z^2 \leqslant R_2^2\} \tag{20}
$$

with uniform density on a unit mass at (x_0, y_0, z_0).