Calculation of triple and multiple integrals

Triple integrals

Example 1. Let A be enclosed by $x=0, y=0, z=0$ and $x+y+z=1$. Calculate

$$\int_A z^2 \, \mathrm{d}(x,y,z). \tag{1}$$

Let $D := \{(x,y)|\, 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1-x\}$. Then

$$\begin{aligned}
\int_A z^2 \, \mathrm{d}(x,y,z) &= \int_D \left[\int_0^{1-x-y} z^2 \, \mathrm{d}z\right] \mathrm{d}(x,y) \\
&= \frac{1}{3}\int_D (1-x-y)^3 \, \mathrm{d}(x,y) \\
&= \frac{1}{3}\int_0^1 \left[\int_0^{1-x} (1-x-y)^2 \, \mathrm{d}y\right] \mathrm{d}x \\
&= \frac{1}{12}\int_0^1 (1-x)^4 \, \mathrm{d}x = \frac{1}{60}.
\end{aligned} \tag{2}$$

Exercise 1. How many times did we apply Fubini? Justify each application of the theorem.

Example 2. Let $A := \left\{(x,y,z)|\, \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leqslant 1\right\}$. Calculate

$$\int_A z^2 \, \mathrm{d}(x,y,z). \tag{3}$$

Let $A_z := \left\{(x,y)|\, \frac{x^2}{a^2\,(1-z^2/c^2)} + \frac{y^2}{b^2\,(1-z^2/c^2)} \leqslant 1\right\}$. We have

$$\begin{aligned}
\int_A z^2 \, \mathrm{d}(x,y,z) &= \int_{-c}^{c} z^2 \left[\int_{A_z} \mathrm{d}(x,y)\right] \mathrm{d}z \\
&= \int_{-c}^{c} z^2 \, \pi \, a \, b \left(1 - \frac{z^2}{c^2}\right) \mathrm{d}z \\
&= \frac{4}{15} \pi \, a \, b \, c^3.
\end{aligned} \tag{4}$$

Exercise 2. Apply symmetry to obtain

$$\int_A (x+y+z)^2 \, \mathrm{d}(x,y,z) \tag{5}$$

using the result in the above example.

Multiple integrals

Example 3. Let $B := \{\boldsymbol{x} \in \mathbb{R}^N|\, \|\boldsymbol{x}\| \leqslant 1\}$. Find its volume.

Denote by V_N the volume. We try to obtain the relation between V_N and V_{N-1}. First recall that, by properties of Jordan measure,

$$\int_{B(\mathbf{0},r)} \mathrm{d}\boldsymbol{x} = r^N \int_B \mathrm{d}\boldsymbol{x} = r^N V_N. \tag{6}$$

$$\begin{aligned} V_N &:= \int_{(x_1^2+\cdots+x_N^2)\leqslant 1} \mathrm{d}(x_1,\ldots,x_N) \\ &= \int_{-1}^{1} \left[\int_{x_1^2+\cdots+x_{N-1}^2 \leqslant 1-x_N^2} \mathrm{d}(x_1,\ldots,x_{N-1}) \right] \mathrm{d}x_N \\ &= \int_{-1}^{1} (1-x_N^2)^{(N-1)/2} V_{N-1}\, \mathrm{d}x_N \\ &= \left[\int_{-1}^{1} (1-x_N^2)^{(N-1)/2} \mathrm{d}x \right] V_{N-1}. \end{aligned} \tag{7}$$

It turns out that

$$\int_{-1}^{1} (1-x_N^2)^{(N-1)/2} \mathrm{d}x = \frac{\Gamma\left(\frac{N+1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{N+2}{2}\right)} \tag{8}$$

where the Gamma function is defined as

$$\Gamma(t) = \int_0^\infty e^{-x} x^{t-1} \mathrm{d}x. \tag{9}$$

It turns out that $\Gamma(1/2) = \sqrt{\pi}$. So the above gives

$$V_N = \frac{\pi^{N/2}}{\Gamma\left(\frac{N+2}{2}\right)}. \tag{10}$$

Example 4. Let $f(x)$ be continuous on $[a,b]$. Prove that

$$\int_a^b \left[\int_a^{x_n} \left[\int_a^{x_{n-1}} \left[\cdots \int_a^{x_2} f(x_1)\, \mathrm{d}x_1 \right] \mathrm{d}x_{n-2} \right] \mathrm{d}x_{n-1} \right] \mathrm{d}x_n = \frac{1}{(n-1)!} \int_a^b f(z)\,(b-z)^{n-1} \mathrm{d}z. \tag{11}$$

Let

$$A := \{(x_1,\ldots,x_n) |\, a \leqslant x_1 \leqslant x_2, a \leqslant x_2 \leqslant x_3, \cdots a \leqslant x_n \leqslant b\}. \tag{12}$$

Now if we define

$$B := \{(x_1,\ldots,x_n) |\, a \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n \leqslant b\} \tag{13}$$

then $A = B$.

Thus we have the integral equals to

$$\begin{aligned} \int_B f(x_1)\, \mathrm{d}\boldsymbol{x} &= \int_a^b \left[\int_{x_1}^b \left[\int_{x_2}^b \left[\cdots \int_{x_{n-1}}^b \mathrm{d}x_n \cdots \right] \mathrm{d}x_3 \right] \mathrm{d}x_2 \right] f(x_1)\, \mathrm{d}x_1 \\ &= \int_a^b \left[\int_{x_1}^b \left[\int_{x_2}^b \left[\cdots \int_{x_{n-2}}^b (b-x_{n-1})\, \mathrm{d}x_{n-1} \cdots \right] \mathrm{d}x_3 \right] \mathrm{d}x_2 \right] f(x_1)\, \mathrm{d}x_1 \\ &= \cdots \\ &= \frac{1}{(n-1)!} \int_a^b f(x_1)\,(b-x_1)^{n-1} \mathrm{d}x_1. \end{aligned} \tag{14}$$

Exercise 3. Let $\Omega := \left\{(x_1,\ldots,x_N) |\, x_i \geqslant 0; \sum_{i=1}^N x_i \leqslant 1\right\}$. Prove that its volume is $1/(n!)$.