Fubini for continuous functions over intervals

We first prove the following theorem for continuous functions.

Theorem 1. *Let $f(\boldsymbol{x})$ be continuous on a compact interval $I = [a, b] \times [c, d]$. Then*

$$\int_{[a,b]\times[c,d]} f(x, y)\,\mathrm{d}(x, y) = \int_a^b \left[\int_c^d f(x, y)\,\mathrm{d}y\right]\mathrm{d}x = \int_c^d \left[\int_a^b f(x, y)\,\mathrm{d}x\right]\mathrm{d}y. \tag{1}$$

Proof. As $f(x,\ y)$ is continuous, for every fixed x_0 and fixed y_0, $f(x_0,\ y)$ and $f(x,\ y_0)$ are continuous. Furthermore,

$$\int_c^d f(x, y)\,\mathrm{d}y \text{ and } \int_a^b f(x, y)\,\mathrm{d}x \tag{2}$$

are also continuous. Thus all the above integrals are well-defined.

We prove

$$\int_{[a,b]\times[c,d]} f(x, y)\,\mathrm{d}(x, y) = \int_a^b \left[\int_c^d f(x, y)\,\mathrm{d}y\right]\mathrm{d}x \tag{3}$$

and the other equality is similar. Wlog assume $a = c = 0, b = d = 1$.

Fix any $\varepsilon > 0$. Since f is continuous on $[0, 1] \times [0, 1]$ it is uniformly continuous and there is $\delta > 0$ such that for any $\|\boldsymbol{x} - \boldsymbol{y}\| < \delta$,

$$|f(\boldsymbol{x}) - f(\boldsymbol{y})| < \varepsilon. \tag{4}$$

Now take $n \in \mathbb{N}$ such that $1/n < \delta/\sqrt{2}$ and divide $[0,\ 1] \times [0,\ 1]$ into squares of the form $I_{ij} := [i\ h,\ (i + 1)\ h) \times [j\,h, (j + 1)\,h)$ for $i, j \in \mathbb{Z}$.

Then we have

$$\forall i, j, \qquad \sup_{I_{ij}} f - \inf_{I_{ij}} f < \varepsilon. \tag{5}$$

Now define

$$g(x, y) := \sup_{I_{ij}} f,\ \ h(x, y) := \inf_{I_{ij}} f, \qquad (x, y) \in I_{ij}. \tag{6}$$

We have

$$\int_I g(x, y) \geqslant \int_I f(x, y) \geqslant \int_I h(x, y), \qquad \int_I g(x, y) - \int_I h(x, y) < \varepsilon. \tag{7}$$

Now it can be checked through direct calculation that

$$\int_I g(x, y) = \int_0^1 \left[\int_0^1 g(x, y)\,\mathrm{d}y\right]\mathrm{d}x, \qquad \int_I h(x, y) = \int_0^1 \left[\int_0^1 h(x, y)\,\mathrm{d}y\right]\mathrm{d}x. \tag{8}$$

Fix $x = x_0$. We have

$$g(x_0, y) \geqslant f(x_0, y) \geqslant h(x_0, y) \tag{9}$$

therefore

$$\int_0^1 g(x_0, y)\,\mathrm{d}y \geqslant \int_0^1 f(x_0, y)\,\mathrm{d}y \geqslant \int_0^1 h(x_0, y)\,\mathrm{d}y \tag{10}$$

for every $x_0 \in [0, 1]$. Consequently

$$\int_0^1 \left[\int_0^1 g(x, y)\,\mathrm{d}y\right] \mathrm{d}x \geqslant \int_0^1 \left[\int_0^1 f(x, y)\,\mathrm{d}y\right] \mathrm{d}x \geqslant \int_0^1 \left[\int_0^1 h(x, y)\,\mathrm{d}y\right] \mathrm{d}x. \tag{11}$$

This gives

$$\left|\int_I f(x, y) - \int_0^1 \left[\int_0^1 f(x, y)\,\mathrm{d}y\right] \mathrm{d}x\right| < \varepsilon \tag{12}$$

and the conclusion follows from the arbitrariness of ε. □

Theorem 2. *Let $I \subseteq \mathbb{R}^N$, $J \subseteq \mathbb{R}^M$ be compact intervals and let $f(\boldsymbol{x}, \boldsymbol{y})$ be continuous on $I \times J$. Then*

$$\int_{I \times J} f(x, y)\,\mathrm{d}(x, y) = \int_I \left[\int_J f(x, y)\,\mathrm{d}y\right] \mathrm{d}x = \int_J \left[\int_I f(x, y)\,\mathrm{d}x\right] \mathrm{d}y. \tag{13}$$

Proof. The proof is similar and is left as exercise. □

Corollary 3. *Let $f(\boldsymbol{x})$ be continuous on $I := [a_1, b_1] \times \cdots \times [a_N, b_N]$. Then we have*

$$\int_I f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_{a_1}^{b_1} \left[\int_{a_2}^{b_2} \left[\cdots\left(\int_{a_N}^{b_N} f(x_1, \ldots, x_N)\,\mathrm{d}x_N\right)\cdots\right]\mathrm{d}x_2\right]\mathrm{d}x_1 \tag{14}$$

and the order of the integration can be arbitrarily changed.

Proof. Exercise. □

Example 4. Let $A := [0, 1] \times [0, 1]$. Calculate

$$\int_A x\,e^{xy}\,\mathrm{d}x\,\mathrm{d}y. \tag{15}$$

Solution. We write

$$\begin{aligned} \int_A x\,e^{xy}\,\mathrm{d}x\,\mathrm{d}y &= \int_0^1 \left[\int_0^1 x\,e^{xy}\,\mathrm{d}y\right] \mathrm{d}x \\ &= \int_0^1 \left[\int_0^x e^z\,\mathrm{d}z\right] \mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 (e^x - 1)\,\mathrm{d}x \\
&= e - 2. \qquad (16)
\end{aligned}
$$

Exercise 1. Let $f \in C^2$. Let $I = [a,b] \times [c,d]$. Calculate

$$\int_I \frac{\partial^2 f}{\partial x \partial y}\,\mathrm{d}x\,\mathrm{d}y. \tag{17}$$

Exercise 2. Calculate the followng.

$$\int_I e^{x+y}\,\mathrm{d}x\,\mathrm{d}y, \qquad I = [0,1]^2; \tag{18}$$

$$\int_I \frac{x^2}{1+y^2}\,\mathrm{d}x\,\mathrm{d}y, \qquad I = [0,1]^2; \tag{19}$$

$$\int_I x \sin(x\,y)\,\mathrm{d}x\,\mathrm{d}y, \qquad I = [0,\pi/2] \times [0,1]. \tag{20}$$

$$\int_I \sin(x+y)\,\mathrm{d}x\,\mathrm{d}y, \qquad I = [0,\pi/2]^2; \tag{21}$$

$$\int_I \sqrt{|y - x^2|}\,\mathrm{d}x\,\mathrm{d}y, \qquad I = [-1,1] \times [0,2]; \tag{22}$$

Exercise 3. Let $I = [0,1]^2$. Calculate $\int_I f(x,y)\mathrm{d}x\,\mathrm{d}y$ for the following $f(x,y)$:

$$
\begin{aligned}
f(x,y) &= \begin{cases} 1 & y \leqslant x^2 \\ 0 & y > x^2 \end{cases}; & (23)\\
f(x,y) &= \begin{cases} 1 - x - y & x + y \leqslant 1 \\ 0 & x + y > 1 \end{cases}; & (24)\\
f(x,y) &= \begin{cases} x + y & x^2 \leqslant y \leqslant 2\,x^2 \\ 0 & \text{elsewhere.} \end{cases}. & (25)
\end{aligned}
$$

Problem 1. (USTC2) Construct $B \subseteq \mathbb{R}^2$ such that the following are satisfied.

1. for every $a \in \mathbb{R}$, $B \cap \{x = a\}$ and $B \cap \{y = a\}$ both consist of at most one single point.
2. $\bar{B} = \mathbb{R}^2$.

Now define

$$f(x,y) = \begin{cases} 1 & (x,y) \in B \\ 0 & \text{elsewhere} \end{cases}. \tag{26}$$

Prove that

a) Both

$$\int_0^1 \left[\int_0^1 f(x,y)\,\mathrm{d}y \right] \mathrm{d}x \text{ and } \int_0^1 \left[\int_0^1 f(x,y)\,\mathrm{d}x \right] \mathrm{d}y \tag{27}$$

exist and equal 0;

b) f is not integrable on $[0,1]^2$.

Fubini: The general case (Optional)

Two-variable Fubini

We still start from the two-variable case.

Theorem 5. *Let $f(x,y): \mathbb{R}^2 \mapsto \mathbb{R}$ be integrable on $I := [a,b] \times [c,d]$. Further assume that for every $x \in [a,b]$, $f(x,y)$ as a function of y is Riemann integrable on $[c,d]$. Then the function*

$$F(x) := \int_c^d f(x,y)\, \mathrm{d}y \tag{28}$$

is Riemann integrable on $[a,b]$ and furthermore

$$\int_I f(x,y)\, \mathrm{d}(x,y) = \int_a^b F(x)\, \mathrm{d}x = \int_a^b \left[\int_c^d f(x,y)\, \mathrm{d}y \right] \mathrm{d}x. \tag{29}$$

If furthermore for every $y \in [c,d]$, $f(x,y)$ as a function of x is Riemann integrable on $[a,b]$, then we can switch the order of integration:

$$\int_a^b \left[\int_c^d f(x,y)\, \mathrm{d}y \right] \mathrm{d}x = \int_c^d \left[\int_a^b f(x,y)\, \mathrm{d}x \right] \mathrm{d}y = \int_I f(x,y)\, \mathrm{d}(x,y). \tag{30}$$

This theorem follows immediately from the following result which reveals what is really going on here.

Theorem 6. *Let $f(x,y): \mathbb{R}^2 \mapsto \mathbb{R}$ and $I := [a,b] \times [c,d]$. Define two functions $\Phi(x)$ and $\phi(x)$ as follows:*

$$\Phi(x) := U(f(x,\cdot),[c,d]), \qquad \phi(x) := L(f(x,\cdot),[c,d]). \tag{31}$$

Here $U(f(x,\cdot,[c,d]))$ and $L(f(x,\cdot),[c,d])$ denote the upper and lower integrals for the function $f(x,y)$ treated as a function of y alone (with x fixed). Then

$$U(\Phi(x),[a,b]) \leqslant U(f(x,y),I); \tag{32}$$
$$L(\phi(x),[a,b]) \geqslant L(f(x,y),I). \tag{33}$$

Exercise 4. Prove Theorem 5 using Theorem 6.

Remark 7. From this theorem we see that two dimensional Riemann integrability puts strong restriction on the behavior of the function along every slice.

Exercise 5. Let $f(x,y): \mathbb{R}^2 \mapsto \mathbb{R}$ be integrable on $I := [a,b] \times [c,d]$. For any $\varepsilon > 0$, Let $S_\varepsilon := \{x \in [a,b] |\ f(x,y)$ as a function of y is not Riemann integrable on $[c,d]$ and $U(f,[c,d]) - L(f,[c,d]) > \varepsilon\}$. Then $\mu_1(S_\varepsilon) = 0$ where μ_1 is the one-dimensional Jordan measure. In other words, if $f(x,y)$ is integrable on I, then most of its "slices" are Riemann integrable.

Remark 8. Note that in the above exercise we cannot replace S_ε by $S := \{x \in [a,b] |\ f(x,y)$ as a function of y is not Riemann integrable on $[c,d]\}$. See the problem below.

Problem 2. Let

$$f(x,y):=\begin{cases}0 & x\in\mathbb{Q}, y\in\mathbb{Q}\\ \frac{1}{p} & x=\frac{r}{p}, (p,r) \text{ co-prime; } y\in\mathbb{Q}^c\\ \frac{1}{q} & x\in\mathbb{Q}^c, y=\frac{s}{q}, (s,q) \text{ co-prime}\\ 0 & x\in\mathbb{Q}^c, y\in\mathbb{Q}^c\end{cases}. \tag{34}$$

Prove that $f(x,y)$ is Riemann integrable on $[0,1]\times[0,1]$. But for every $x\in[0,1]\cap\mathbb{Q}$, $f(x,y)$ is not Riemann integrable on $[0,1]$.

Proof. (of Theorem 6) Recall our results regarding "uniform partition". For any $n\in\mathbb{N}$, set $h_1:=\frac{b-a}{n}$ and $h_2:=\frac{d-c}{n}$. Let $I_{ij,h}:=[a+(i-1)\,h_1, a+i\,h_1]\times[c+(j-1)\,h_2, c+j\,h_2]$. Then we know that

$$\lim_{n\to\infty}\sum_{i,j=1}^n f_{ij}\,h_1\,h_2=\lim_{n\to\infty}\sum_{i,j=1}^n F_{ij}\,h_1\,h_2=\int_I f(x,y)\,\mathrm{d}(x,y) \tag{35}$$

where

$$f_{ij}:=\inf_{(x,y)\in I_{ij,h}} f(x,y),\qquad F_{ij}:=\sup_{(x,y)\in I_{ij,h}} f(x,y). \tag{36}$$

Now for each (i,j), we have

$$\forall (x,y)\in I_{ij,h},\qquad f_{ij}\leqslant f(x,y)\Longrightarrow f_{ij}\,h_2\leqslant L(f(x,y),[c+(j-1)\,h_2, c+j\,h_2]); \tag{37}$$

Since this is true for all $x\in[a+(i-1)\,h_1, a+i\,h_1]$, we have

$$f_{ij}\,h_1\,h_2\leqslant L(L(f(x,y),[c+(j-1)\,h_2,c+j\,h_2]),[a+(i-1)\,h_1,a+i\,h_1]). \tag{38}$$

Now summing over i,j we have

$$\sum_{i,j=1}^n f_{ij}\,h_1\,h_2\leqslant L(\phi(x),[a,b]). \tag{39}$$

The other inequality can be proved similarly. □

Exercise 6. Let $f\colon\mathbb{R}\mapsto\mathbb{R}$, $a<b<c$. Prove that

$$L(f,[a,b])+L(f,[b,c])=L(f,[a,c]). \tag{40}$$

General cases

The proof for the general case is similar.

Theorem 9. (Fubini) *Let $f(\boldsymbol{x},\boldsymbol{y})$ ($\boldsymbol{x}\in\mathbb{R}^M,\boldsymbol{y}\in\mathbb{R}^N$) be integrable on $I:=I_1\times I_2$ where $I_1\subseteq\mathbb{R}^M, I_2\subseteq\mathbb{R}^N$. Assume that for every $\boldsymbol{x}\in I_1$ the function $f(\boldsymbol{x},\boldsymbol{y})$ as a function of $\boldsymbol{y}$ only is integrable on I_2, then*

$$\int_I f(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}(\boldsymbol{x},\boldsymbol{y})=\int_{I_1}\left[\int_{I_2} f(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right]\mathrm{d}\boldsymbol{x}. \tag{41}$$

If furthermore for every $\boldsymbol{y}\in I_2$ the function $f(\boldsymbol{x},\boldsymbol{y})$ as a function of $\boldsymbol{x}$ only is integrable on I_1, then

$$\int_{I_1}\left[\int_{I_2} f(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right]\mathrm{d}\boldsymbol{x}=\int_{I_2}\left[\int_{I_1} f(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\right]\mathrm{d}\boldsymbol{y}=\int_I f(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}(\boldsymbol{x},\boldsymbol{y}). \tag{42}$$

Exercise 7. Let $A:=\{(x,y,z)|\,x^2+y^2+z^2\leqslant 1\}$. Prove that

$$\mu(A)=\int_{-1}^1\left[\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\left(\int_{-\sqrt{1-z^2-y^2}}^{\sqrt{1-z^2-y^2}} 1\,\mathrm{d}x\right)\mathrm{d}y\right]\mathrm{d}z. \tag{43}$$