Further study of Riemann integrability

Uniform partition

In some situations it is beneficial to restrict ourselves to a special class of simple functions.

Definition 1. (Uniform partition) *A uniform partition of size $h > 0$, denoted P_h, is the collection of the following compact intervals:*

$$P_h := \{[i_1 h, (i_1 + 1) h] \times \cdots \times [i_N h, (i_N + 1) h] | (i_1, \ldots, i_N) \in \mathbb{Z}^N\}. \tag{1}$$

Theorem 2. *Let $A \subseteq \mathbb{R}^N$ be Jordan measurable. For every $h > 0$ denote by $n_1(h)$ the number of intervals in P_h contained in A^o, and $n_2(h)$ the number of intervals in P_h with non-empty intersection with A. Then*

$$\mu(A) = \inf_{h>0} [n_2(h) h^N] = \sup_{h>0} [n_1(h) h^N] = \lim_{h \to 0} [n_2(h) h^N] = \lim_{h \to 0} [n_1(h) h^N]. \tag{2}$$

Proof. Take any $\varepsilon > 0$. Since A is Jordan measurable, there are simple graphs B, C such that

$$B \subseteq A^o, \bar{A} \subseteq C, \qquad \mu(A) - \frac{\varepsilon}{2} \leqslant \mu(B) \leqslant \mu(A) \leqslant \mu(C) \leqslant \mu(A) + \frac{\varepsilon}{2}. \tag{3}$$

Now consider $B_h := \cup_{I \in P_h, I \subseteq B} I$. Let $m_1(h)$ denote the number of intervals in B_h. Then clearly $m_1(h) \leqslant n_1(h)$. Furthermore we have

$$m_1(h) h^N = \mu(B_h) \geqslant \mu(B) - h L \tag{4}$$

where L is the total length of the boundary of B (note that as B is a simple graph we do not need any calculus to define L). Taking $h < \frac{\varepsilon}{2 L}$ we see that

$$\mu(A) \geqslant n_1(h) h^N \geqslant m_1(h) h^N \geqslant \mu(A) - \varepsilon. \tag{5}$$

Similarly we have, when $h < h_0$ for some h_0 determined by ε,

$$\mu(A) \leqslant n_2(h) h^N \leqslant \mu(A) + \varepsilon. \tag{6}$$

Thus by definition (2) is true. □

Theorem 3. *Let f be a simple function and let A be Jordan measurable. Let*

$$W_{h,\text{in}} := \{g \leqslant f | \ g \text{ is constant on } I^o \text{ for every } I \in P_h\}; \tag{7}$$

$$W_{h,\text{out}} := \{h \geqslant f | \ h \text{ is constant on } I^o \text{ for every } I \in P_h\}. \tag{8}$$

Then

$$\int_A f(x) \, \mathrm{d}x = \lim_{h \longrightarrow 0} \left[\sup_{g \in W_{h,\text{in}}} \int_A g(x) \, \mathrm{d}x \right] = \lim_{h \to 0} \left[\inf_{h \in W_{h,\text{in}}} \int_A h(x) \, \mathrm{d}x \right]. \tag{9}$$

Proof. Since f is an integrable simple function, $f^+ := \max\{f,0\}$ and $f^- := \min\{f,0\}$ are both integrable simple functions. Therefore we only need to prove the above for non-negative simple functions. By definition of simple functions

$$f^+ = \sum_{i=1}^{n} c_i \, 1_{A_i}(x) \tag{10}$$

where $c_i \geqslant 0$. So it suffices to prove the theorem for $1_{A_i}(x)$, which is done in Theorem 2. □

Riemann integrability using uniform partition

Theorem 4. *Let* $f: \mathbb{R}^N \mapsto \mathbb{R}$. *Define*

$$W_{h,\text{in}}(f) := \{g \leqslant f \mid g \text{ is constant on } I^o \text{ for every } I \in P_h\}; \tag{11}$$

$$W_{h,\text{out}}(f) := \{h \geqslant f \mid h \text{ is constant on } I^o \text{ for every } I \in P_h\}. \tag{12}$$

Then f *is integrable if and only if*

$$\lim_{h \longrightarrow 0} \left[\sup_{g \in W_{h,\text{in}}} \int_A g(x)\, \mathrm{d}x \right] = \lim_{h \rightarrow 0} \left[\inf_{h \in W_{h,\text{in}}} \int_A h(x)\, \mathrm{d}x \right] \tag{13}$$

and in this case the common value is $\int_A f(x)\,\mathrm{d}x$.

Proof. First clearly all functions in $W_{h,\text{in}}(f)$ and $W_{h,\text{out}}(f)$ are simple functions. Therefore the equality of the two limits indicates the integrability of f by definition.

On the other hand, if f is integrable, then for any $\varepsilon > 0$ there are simple functions g, h such that

$$g \leqslant f \leqslant h, \qquad \int_A f(x)\,\mathrm{d}x - \frac{\varepsilon}{2} \leqslant \int_A g(x)\,\mathrm{d}x \leqslant \int_A f(x)\,\mathrm{d}x \leqslant \int_A h(x)\,\mathrm{d}x \leqslant \int_A f(x)\,\mathrm{d}x + \frac{\varepsilon}{2}. \tag{14}$$

By Theorem 3 there is $h_0 > 0$ such that for all $0 < h < h_0$,

$$\int_A f(x)\,\mathrm{d}x - \varepsilon \leqslant \int_A g(x)\,\mathrm{d}x - \frac{\varepsilon}{2} \leqslant \sup_{u \in W_{h,\text{in}}(g)} \int_A u(x)\,\mathrm{d}x \tag{15}$$

and

$$\inf_{v \in W_{h,\text{out}}(h)} \int_A v(x)\,\mathrm{d}x \leqslant \int_A h(x)\,\mathrm{d}x + \frac{\varepsilon}{2} \leqslant \int_A f(x)\,\mathrm{d}x + \varepsilon. \tag{16}$$

Now note that

$$W_{h,\text{in}}(g) \subseteq W_{h,\text{in}}(f) \Longrightarrow \sup_{u \in W_{h,\text{in}}(g)} \int_A u(x)\,\mathrm{d}x \leqslant \sup_{u \in W_{h,\text{in}}(f)} \int_A u(x)\,\mathrm{d}x \tag{17}$$

similarly

$$\inf_{v \in W_{h,\text{out}}(h)} \int_A v(x)\,\mathrm{d}x \geqslant \inf_{v \in W_{h,\text{out}}(f)} \int_A v(x)\,\mathrm{d}x. \tag{18}$$

On the other hand we have

$$\sup_{u \in W_{h,\text{in}}(f)} \int_A u(x)\,\mathrm{d}x \leqslant \int_A f(x)\,\mathrm{d}x \leqslant \inf_{v \in W_{h,\text{out}}(f)} \int_A v(x)\,\mathrm{d}x. \tag{19}$$

Putting the above together we have for all $0<h<h_0(\varepsilon)$,

$$\int_A f(x)\,\mathrm{d}x-\varepsilon\leqslant\sup_{u\in W_{h,\text{in}}(f)}\int_A u(x)\,\mathrm{d}x\leqslant\int_A f(x)\,\mathrm{d}x \tag{20}$$

and

$$\int_A f(x)\,\mathrm{d}x\leqslant\inf_{v\in W_{h,\text{out}}(f)}\int_A v(x)\,\mathrm{d}x\leqslant\int_A f(x)\,\mathrm{d}x+\varepsilon. \tag{21}$$

By defintition this means both limits

$$\lim_{h\longrightarrow 0}\left[\sup_{g\in W_{h,\text{in}}}\int_A g(x)\,\mathrm{d}x\right],\qquad \lim_{h\rightarrow 0}\left[\inf_{h\in W_{h,\text{in}}}\int_A h(x)\,\mathrm{d}x\right] \tag{22}$$

exist and both equal $\int_A f(x)\,\mathrm{d}x$. □

Exercise 1. Fill in the details of the above proof.

Regular domain

In practice it is often advantageous to make the integratiom domain E "regular", such as interval or ball. The following theorem discusses this possibility.

Theorem 5. *Let $f\colon\mathbb{R}^N\mapsto\mathbb{R}$, $E_1\subseteq E_2\subseteq\mathbb{R}^N$. If*

i. E_1 is Jordan measurable;

ii. f is Riemann integrable on E_1.

Then the following function:

$$\tilde{f}(\boldsymbol{x}):=\begin{cases} f(\boldsymbol{x}) & \boldsymbol{x}\in E_1\\ 0 & \boldsymbol{x}\notin E_1\end{cases} \tag{23}$$

is Riemann integrable on E_2 with

$$\int_{E_2}\tilde{f}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}=\int_{E_1} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}. \tag{24}$$

Proof. Take any $\varepsilon>0$. Since f is integrable on E_1 there are simple functions $g\geqslant f\geqslant h$ such that

$$\int_{E_1} g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}-\int_{E_1} h(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}<\varepsilon. \tag{25}$$

Now we define

$$\tilde{g}(\boldsymbol{x}):=\begin{cases} g(\boldsymbol{x}) & \boldsymbol{x}\in E_1\\ 0 & \boldsymbol{x}\notin E_1\end{cases},\qquad \tilde{h}(\boldsymbol{x}):=\begin{cases} h(\boldsymbol{x}) & \boldsymbol{x}\in E_1\\ 0 & \boldsymbol{x}\notin E_1\end{cases} \tag{26}$$

Clearly $\tilde{g},\tilde{h}$ are still simple functions and furthermore satisfy $\tilde{g}\geqslant\tilde{f}\geqslant\tilde{h}$. But

$$\int_{E_2}\tilde{g}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}=\int_{E_1} g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x},\qquad \int_{E_2}\tilde{h}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}=\int_{E_1} h(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}. \tag{27}$$

Consequently

$$\int_{E_1}\tilde{g}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}-\int_{E_1}\tilde{h}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}<\varepsilon. \tag{28}$$

This gives the integrability of $\tilde{f}$ on E_2 as well as (24). □

Exercise 2. If $\tilde{f}$ is integrable on E_2, can we conclude f is integrable on E_1?

Remark 6. Thanks to the above theorem, when calculating higher dimensional integrals, we can always take the domain to be a compact interval.