Riemann Integrability

Lemma 1. *Let E_1, E_2 be Jordan measurable with $E_1 \subseteq E_2$. Then if f is integrable on E_2 it is also integrable on E_1.*

Proof. Since f is integrable on E_2 there are simple functions $g_n \geqslant f$ and $h_n \leqslant f$ such that

$$\lim_{n \longrightarrow \infty} \int_{E_2} [g_n(x) - h_n(x)]\, \mathrm{d}x = 0. \tag{1}$$

Now we have

$$\begin{aligned} 0 \leqslant \int_{E_1} [g_n(x) - h_n(x)]\, \mathrm{d}x &= \int_{E_2} [g_n(x) - h_n(x)]\, \mathrm{d}x - \int_{E_2 - E_1} [g_n(x) - h_n(x)]\, \mathrm{d}x \\ &\leqslant \int_{E_2} [g_n(x) - h_n(x)]\, \mathrm{d}x \end{aligned} \tag{2}$$

Application of Squeeze Theorem gives

$$\lim_{n \longrightarrow \infty} \int_{E_2} [g_n(x) - h_n(x)]\, \mathrm{d}x = 0 \tag{3}$$

and integrability follows. □

Theorem 2. *Let A be Jordan measurable and $f: \mathbb{R}^N \mapsto \mathbb{R}$ be continuous on $\bar{A}$. Then f is integrable on A.*

Proof. Since A is Jordan measurable it must be bounded and consequently $\bar{A}$ is bounded and closed. By Heine-Borel $\bar{A}$ is compact. As a consequence f is uniformly continuous on $\bar{A}$.

Take any $\varepsilon > 0$. Then there is $\delta > 0$ such that whenever $x, y \in \bar{A}$ with $\|x - y\| < \delta$, there holds

$$|f(x) - f(y)| < \frac{\varepsilon}{\mu(A)}. \tag{4}$$

Now set $h := \delta / \sqrt{2}$ and consider the intersection of A with intervals of the form

$$I := [i_1 h, (i_1 + 1)\, h) \times \cdots \times [i_N h, (i_N + 1)\, h) \tag{5}$$

where $i_1, \ldots, i_N \in \mathbb{Z}$. Then we see that A can be written as a union

$$A = \cup_{i=1}^{m} A_i \tag{6}$$

where each A_i is a subset of an interval of the above form, and furthermore $A_i \cap A_j = \varnothing$ when $i \neq j$. Now define

$$g(x) = \sup_{x \in A_i} f(x), \qquad x \in A_i; \qquad h(x) = \inf_{x \in A_i} f(x), \qquad x \in A_i \tag{7}$$

Clearly $g \geqslant f \geqslant h$ and g, h are simple functions. Furthermore we have

$$\forall x, y \in A_i, \qquad \|x - y\| < \delta \tag{8}$$

so

$$g(x) - h(x) < \frac{\varepsilon}{\mu(A)}. \tag{9}$$

This gives

$$\int_A [g(x) - h(x)]\,\mathrm{d}x < \varepsilon \tag{10}$$

and integrability follows. □

Theorem 3. *Let $A \subseteq \mathbb{R}^N$ be Jordan measurable and $f: \mathbb{R}^N \mapsto \mathbb{R}$ be bounded. Denote*

$$S := \{x \in \mathbb{R}^N |\ f(x) \text{ is not continuous at } x\}. \tag{11}$$

Then

$$\mu(S \cap A) = 0 \Longrightarrow f(x) \text{ is integrable on } A. \tag{12}$$

Proof. Since $f(x)$ is bounded, assume $|f(x)| \leqslant M \in \mathbb{R}$. Furthermore as A is Jordan measurable, $\mu(\partial A) = 0$. Let $T := (S \cap A) \cup \partial A$. We have $\mu(T) = 0$.

For any $\varepsilon > 0$, since $\mu(T) = 0$, there is a simple graph $E \subseteq \mathbb{R}^N$ such that

$$\bar{T} \subseteq E \text{ and } \mu(E) < \frac{\varepsilon}{4\,M}. \tag{13}$$

Enlarging E a little bit we can further require

$$\bar{T} \subseteq E^o. \tag{14}$$

Now clearly f is continuous on $\Omega := \bar{A} - E^o = A - E^o$ and therefore is integrable on $A - E$. Thus there are simple functions g, h such that $g \geqslant f \geqslant h$ on $A - E$ and

$$\int_{A-E} [g(x) - h(x)]\,\mathrm{d}x < \frac{\varepsilon}{2}. \tag{15}$$

Now define

$$u(x) := \begin{cases} g(x) & x \in A - E \\ M & x \in E \end{cases}, \qquad v(x) := \begin{cases} h(x) & x \in A - E \\ -M & x \in E \end{cases}. \tag{16}$$

We have $u \geqslant f \geqslant v$ on A and furthermore

$$\int_A [u(x) - v(x)]\,\mathrm{d}x < \varepsilon. \tag{17}$$

Therefore f is integrable on A. □

Example 4. The Riemann function

$$f(x) := \begin{cases} 1/q & x = p/q \text{ with } p, q \text{ co-prime} \\ 0 & x \in [0,1] - \mathbb{Q} \end{cases} \tag{18}$$

is Riemann integrable but its discontinuity does not have zero Jordan measure.

Theorem 5. *Let $A \subseteq \mathbb{R}^N$ be Jordan measurable and $f: \mathbb{R}^N \mapsto \mathbb{R}$ be bounded. Let*

$$S := \{x \in \mathbb{R}^N |\ f(x) \text{ is not continuous at } x\}. \tag{19}$$

Then

$$S = \cup_{n=1}^{\infty} S_n \tag{20}$$

where

$$S_n := \left\{ x \in \mathbb{R}^N | \operatorname{osc}(f, x) > \frac{1}{n} \right\}. \tag{21}$$

Here the oscillation of f is defined as

$$\operatorname{osc}(f, x) := \lim_{r \searrow 0} \left[\sup_{y \in B(x,r)} f(y) - \inf_{y \in B(x,r)} f(y) \right]. \tag{22}$$

Then f is Riemann integrable if $\mu(S_n) = 0$ for every $n \in \mathbb{N}$.

Proof. Left as exercise. □

Remark 6. The above theorem reveals one problem with our Jordan-Darboux-Riemann integration theory. Also note that the theorem is not "if and only if". It turns out that f is Riemann integrable if the Lebesgue measure of the set of its discontinuities is zero.

Question 7. *Compare the above with the following theorem by Lebesgue: f is Riemann integrable if and only if the set of its discontinuous points has Lebesgue measure 0.*

Exercise 1. Let $A \subseteq \mathbb{R}^N$ be Jordan measurable. Let $f, g: \mathbb{R}^N \mapsto \mathbb{R}$ be integrable on A and such that

$$\mu(A \cap \{x \in \mathbb{R}^N | f(x) \neq g(x)\}) = 0. \tag{23}$$

Prove that

$$\int_A f(x)\, \mathrm{d}x = \int_A g(x)\, \mathrm{d}x. \tag{24}$$

Exercise 2. (USTC2) Let $f: \mathbb{R}^N \mapsto \mathbb{R}$ be integrable on an interval I with $\int_I f(x)\, \mathrm{d}x > 0$. Prove that there is an interval $J \subseteq I$ such that $f > 0$ on J.

Exercise 3. Is the following function integrable on $[0, 1] \times [0, 1]$?

$$f(x, y) := \begin{cases} \sin\left(\frac{1}{x\,y}\right) & x\,y \neq 0 \\ 0 & x\,y = 0 \end{cases}. \tag{25}$$

Exercise 4. Let $f: \mathbb{R}^N \mapsto \mathbb{R}$ be integrable on a measurable set A. Then so does $|f|$. Does the converse hold? Justify.

Exercise 5. Let A be Jordan measurable and $f_n: \mathbb{R}^N \mapsto \mathbb{R}$ be continuous on A. Further assume that for every $x \in A$, $f_n(x)$ is decreasing and $\lim_{n \to \infty} f_n(x) = 0$. Prove

$$\lim_{n \to \infty} \int_A f_n(x)\, \mathrm{d}x = 0. \tag{26}$$

Can we drop continuity or decreasing? Justify your answers.