## Definition of Riemann Integration

**Definition 1.** *Let $f: \mathbb{R}^N \mapsto \mathbb{R}$. Let $E \subseteq \mathbb{R}^N$ be bounded. Denote*

$$W_{\text{upper}}(f) := \{g \geqslant f, g \text{ is a simple function}\}; \quad W_{\text{lower}}(f) := \{h \leqslant f, h \text{ is a simple function}\}. \tag{1}$$

*Then define the upper and lower integrals of $f$ on $E$ as:*

$$U(f, E) := \inf_{g \in W_{\text{upper}}} \int_E g(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}; \quad L(f, E) := \sup_{h \in W_{\text{lower}}} \int_E h(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}. \tag{2}$$

*We say $f(x)$ is Riemann integrable on the set $E$ if and only if $U(f, E) = L(f, E)$ is finite. We denote this common value by*

$$\int_E f(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}. \tag{3}$$

**Exercise 1.** Let $f: \mathbb{R}^N \mapsto \mathbb{R}$. Let $E \subseteq \mathbb{R}^N$ be bounded. Then $U(f, E) \geqslant L(f, E)$.

**Theorem 2.** *Let $f: \mathbb{R}^N \mapsto \mathbb{R}$ be Riemann integrable on $E$, then it is bounded on $E$. That is there is $M > 0$ such that $\forall \boldsymbol{x} \in E$, $|f(\boldsymbol{x})| \leqslant M$.*

**Proof.** Assume the contrary. Then either $f$ is not bounded above or not bounded below. Wlog assume $f$ is not bounded above.

Take any $g \in W_{\text{upper}}(f)$. $g$ is a simple function so

$$g(\boldsymbol{x}) = \sum_{i=1}^{n} c_i\, 1_{A_i}(x). \tag{4}$$

It is clear that

$$g(\boldsymbol{x}) \leqslant \sum_{i=1}^{n} |c_i| < \infty \tag{5}$$

is bounded above. So $W_{\text{upper}}(f)$ is empty and $U(f, E)$ is not finite. □

**Theorem 3.** *Let $f: \mathbb{R}^N \mapsto \mathbb{R}$ be bounded. Let $E \subseteq \mathbb{R}^N$ be such that $\mu(E) = 0$. Then $f$ is integrable on $E$ with $\int_E f(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = 0$.*

**Proof.** As $f$ is bounded, there is $M > 0$ such that $|f(\boldsymbol{x})| \leqslant M$, that is $-M \leqslant f(\boldsymbol{x}) \leqslant M$. Now take simple functions

$$g(\boldsymbol{x}) = M, \qquad h(\boldsymbol{x}) = -M. \tag{6}$$

It is clear that $g \in W_{\text{upper}}(f)$ and $h \in W_{\text{lower}}(f)$. Furthermore we have

$$\int_E g(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = \int_E h(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = 0. \tag{7}$$

Therefore

$$0 \leqslant L(f, E) \leqslant U(f, E) \leqslant 0. \tag{8}$$

Consequently $U(f,E)=L(f,E)=0$. □

**Theorem 4.** *Let $E\subseteq\mathbb{R}^N$. Let $f:\mathbb{R}^N\mapsto\mathbb{R}$. Then $f$ is integrable on $E$ if and only if there are simple functions $g_n\geqslant f$ and $h_n\leqslant f$ such that*

$$\lim_{n\longrightarrow\infty}\int_E (g_n-h_n)\,\mathrm{d}\boldsymbol{x}=0. \tag{9}$$

**Proof.**

- If. We have

$$\int_E (g_n(\boldsymbol{x})-h_n(\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x}\geqslant U(f,E)-L(f,E) \tag{10}$$

  and the conclusion follows.

- Only if. By definitions of sup and inf, for every $n\in\mathbb{N}$ there are $g_n\geqslant f$ and $h_n\leqslant f$, simple functions, such that

$$\int_E g_n(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\leqslant U(f,E)+\frac{1}{n},\qquad \int_E h_n(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\geqslant L(f,E)-\frac{1}{n}. \tag{11}$$

  As $f$ is integrable

$$U(f,E)=L(f,E), \tag{12}$$

  we have

$$0\leqslant\int_E (g_n-h_n)\,\mathrm{d}\boldsymbol{x}\leqslant\frac{2}{n}. \tag{13}$$

  Application of the Squeeze Theorem gives the desired result. □

The above theorem can be written in a slightly different way.

**Theorem 5.** *Let $E\subseteq\mathbb{R}^N$. Let $f:\mathbb{R}^N\mapsto\mathbb{R}$. Then $f$ is integrable on $E$ if and only if for every $\varepsilon>0$ there are simple functions $g\geqslant f\geqslant h$ such that*

$$\int_E [g(\boldsymbol{x})-h(\boldsymbol{x})]\,\mathrm{d}\boldsymbol{x}<\varepsilon. \tag{14}$$

**Example 6.** Prove that $f(x,y)=\sin(x+y)$ is integrable on $[0,1]\times[0,1]$.

**Proof.** For any $n\in\mathbb{N}$, set

$$g_n(x,y):=\begin{cases}\max\limits_{\frac{i}{n}\leqslant x\leqslant\frac{i+1}{n},\frac{j}{n}\leqslant y\leqslant\frac{j+1}{n}}\sin(x+y) & \dfrac{i}{n}<x<\dfrac{i+1}{n},\dfrac{j}{n}<y<\dfrac{j+1}{n}\\ 1 & \text{elsewhere}\end{cases},i,j\in\{0,\ldots,n-1\} \tag{15}$$

$$h_n(x,y):=\begin{cases}\min\limits_{\frac{i}{n}\leqslant x\leqslant\frac{i+1}{n},\frac{j}{n}\leqslant y\leqslant\frac{j+1}{n}}\sin(x+y) & \dfrac{i}{n}<x<\dfrac{i+1}{n},\dfrac{j}{n}<y<\dfrac{j+1}{n}\\ -1 & \text{elsewhere}\end{cases},i,j\in\{0,\ldots,n-1\} \tag{16}$$

Then $h_n, g_n$ are simple functions and $h_n \leqslant f \leqslant g_n$.

Now for any $(x_1, y_1), (x_2, y_2) \in \left[\frac{i}{n}, \frac{i+1}{n}\right] \times \left[\frac{j}{n}, \frac{j+1}{n}\right]$, we have

$$|\sin(x_1 + y_1) - \sin(x_2 + y_2)| = |\cos(\xi)|\,|(x_1 + y_1) - (x_2 + y_2)| \leqslant \frac{2}{n}. \tag{17}$$

Therefore for all $(x, y) \in \left(\frac{i}{n}, \frac{i+1}{n}\right) \times \left(\frac{j}{n}, \frac{j+1}{n}\right)$,

$$g_n(x, y) - h_n(x, y) \leqslant \frac{2}{n}. \tag{18}$$

Consequently

$$\int_E (g_n - h_n)\,\mathrm{d}\boldsymbol{x} \leqslant \sum_{i=1, j=1}^{n} \frac{2}{n} \cdot \frac{1}{n^2} = \frac{2}{n} \longrightarrow 0 \tag{19}$$

Integrability now follows. □

**Exercise 2.** Prove that $\sin(x + y)$ is integrable on $\Omega := \{(x, y) |\, x^2 + y^2 \leqslant 1\}$.

**Exercise 3.** Consider calculating $\int_\Omega \sin(x + y)\,\mathrm{d}(x, y)$ as follows: For any $h > 0$, define

$$I(h) := \sum_{i, j \in \mathbb{Z};\, (ih, jh) \in \Omega} \sin(i\,h + j\,h)\,h^2. \tag{20}$$

For what $h$ can we be sure that

$$\left| I(h) - \int_\Omega \sin(x + y)\,\mathrm{d}(x, y) \right| < 10^{-3}? \tag{21}$$

**Exercise 4.** Let $I := [a, b] \times [c, d]$. Let $f(x) \colon [a, b] \mapsto \mathbb{R}$, $g(x) \colon [c, d] \mapsto \mathbb{R}$ . Let $F(x, y) := f(x)\,g(y)$. Prove that $F(x, y)$ is integrable on $I$ if and only if $f, g$ are integrable on $[a, b], [c, d]$ respectively. Furthermore we have

$$\int_I F(x, y)\,\mathrm{d}(x, y) = \left[\int_a^b f(x)\,\mathrm{d}x\right] \left[\int_c^d g(x)\,\mathrm{d}x\right]. \tag{22}$$

Similar to simple functions, we have the following properties.

**Theorem 7.** *Let $E \subseteq \mathbb{R}^N$. Let $f, g$ be Riemann integrable on $E$. Then*

*a) For every $c \in \mathbb{R}$, $c\,f$ is Riemann integrable on $E$, and*

$$\int_E c\,f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = c \int_A f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}. \tag{23}$$

*b) $f \pm g$ is Riemann integrable on $E$, with*

$$\int_E (f \pm g)(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_E f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \pm \int_E g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}. \tag{24}$$

*c) If $f \geqslant g$ for all $x \in E$, then*

$$\int_E f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \geqslant \int_E g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}. \tag{25}$$

**Proof.** We prove b) and leave a),c) for exercises.

As $f, g$ are integrable on $A$, we can find simple functions $u_n \geqslant f \geqslant v_n$, $w_n \geqslant g \geqslant h_n$, such that

$$\lim_{n \to \infty} \int_A (u_n(x) - v_n(x)) \, \mathrm{d}x = 0, \qquad \lim_{n \longrightarrow \infty} \int_A (w_n(x) - h_n(x)) \, \mathrm{d}x = 0. \tag{26}$$

Now clearly

$$u_n + w_n \geqslant f + g \geqslant v_n + h_n \tag{27}$$

and

$$\int_A [(u_n + w_n) - (v_n + h_n)] \, \mathrm{d}x = \int_A (u_n - v_n) \, \mathrm{d}x + \int_A (w_n - h_n) \, \mathrm{d}x \tag{28}$$

and the conclusion follows. □

**Theorem 8.** *Let $f$ be integrable on $E_1$ and also on $E_2$. Then $f$ is integrable on $E_1 \cap E_2, E_1 \cup E_2, E_1 - E_2$. Furthermore*

$$\int_{E_1 \cup E_2} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = \int_{E_1} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} + \int_{E_2} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} - \int_{E_1 \cap E_2} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}. \tag{29}$$

*In particular when $\mu(E_1 \cap E_2) = 0$ we have*

$$\int_{E_1 \cup E_2} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = \int_{E_1} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} + \int_{E_2} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}. \tag{30}$$

**Proof.** Left as exercises. □

**Theorem 9. (MVT)** *Let $f, g$ be integrable on $E$. Denote*

$$M := \sup_{\boldsymbol{x} \in E} f(\boldsymbol{x}), \qquad m := \inf_{\boldsymbol{x} \in E} f(\boldsymbol{x}). \tag{31}$$

*Assume $g \geqslant 0$. Then there is $c \in [m, M]$ such that*

$$\int_E f(\boldsymbol{x}) \, g(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = c \int_E g(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}. \tag{32}$$

**Proof.** Exercise. □

**Remark 10.** If we further assume $f$ is continuous on $\bar{E}$ and $\bar{E}$ is connected (doesn't need to be path connected), we can take $c = f(\boldsymbol{\xi})$ for some $\boldsymbol{\xi} \in \bar{E}$.