Integration of simple functions

For any Jordan measurable set $A \subseteq \mathbb{R}^N$, we define

$$\int_A 1 \, \mathrm{d}\boldsymbol{x} := I(1, A) := \mu(A). \tag{1}$$

To extend this definition to more complicated functions, we introduce the idea of "characteristic function".

Definition 1. (Characteristic function) *A function $f: \mathbb{R}^N \mapsto \mathbb{R}$ is said to be the characteristic function of a set $A \subseteq \mathbb{R}$, denoted $f(\boldsymbol{x}) = 1_A(\boldsymbol{x})$, if*

$$f(\boldsymbol{x}) = \begin{cases} 1 & \boldsymbol{x} \in A \\ 0 & \boldsymbol{x} \notin A \end{cases}. \tag{2}$$

Definition 2. (Integration of characteristic functions) *Let A, $E \subseteq \mathbb{R}^N$. We say the characteristic function $1_A(\boldsymbol{x})$ is integrable on E if and only if $A \cap E$ is Jordan measurable. We define the integral to be*

$$\int_E 1_A(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} := \mu(A \cap E). \tag{3}$$

Definition 3. (Simple functions) *A function $f: \mathbb{R}^N \mapsto \mathbb{R}$ is said to be a "simple function" if there are $c_1, \ldots, c_n \in \mathbb{R}$ and $A_1, \ldots, A_n \subseteq \mathbb{R}^N$, such that*

$$f(\boldsymbol{x}) = \sum_{i=1}^n c_i \, 1_{A_i}(\boldsymbol{x}). \tag{4}$$

Theorem 4. *Let f, g be simple functions and $c \in \mathbb{R}$. Then the following are also simple functions:*

$$|f|, \ c f, f \pm g, f g. \tag{5}$$

Proof. Left and exercise. ☐

Theorem 5. *Let $f: \mathbb{R}^N \mapsto \mathbb{R}$ be a simple function. Then there is a unique set of numbers $c_1, \ldots, c_n \in \mathbb{R}$ with $c_i \neq 0$, $c_i \neq c_j$ for every $i \neq j$ and sets $A_1, \ldots, A_n \subseteq \mathbb{R}^N$ with $A_i \cap A_j = \varnothing$ for every $i \neq j$, such that $f(\boldsymbol{x}) = \sum_{i=1}^n c_i \, 1_{A_i}(\boldsymbol{x})$.*

Proof. By definition the image $f(\mathbb{R}^N)$ is a finite set of real numbers. We denote it by $\{c_1, \ldots, c_n\}$. Now define $A_i = f^{-1}(\{c_i\})$. That we can make $c_i \neq 0$ is obvious. ☐

Definition 6. (Integration of simple functions) *Let $f(\boldsymbol{x}) = \sum_{i=1}^n c_i \, 1_{A_i}(\boldsymbol{x})$ be a simple function with c_i, A_i satisfying $c_i \neq 0$ for every i, $c_i \neq c_j, A_i \neq A_j$ for every $i \neq j$. Then f is integrable on $E \subseteq \mathbb{R}^N$ if and only if*

$$\forall i, \qquad A_i \cap E \text{ is Jordan measurable.} \tag{6}$$

We define

$$\int_E f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} := \sum_{i=1}^n c_i \, \mu(A_i \cap E). \tag{7}$$

Exercise 1. Let $E \subseteq \mathbb{R}^N$ be such that $\mu(E) = 0$. Let f be a simple function. Prove that f is integrable on E with $\int_E f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0$.

Exercise 2. Let $E \subseteq \mathbb{R}^N$ and let $f \geqslant 0$ be a simple function integrable on E. Prove that $\int_E f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \geqslant 0$. Further prove that

$$\int_E f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0 \Longleftrightarrow f(\boldsymbol{x}) = 0. \tag{8}$$

We can easily prove the following.

Theorem 7. *Let $E \subseteq \mathbb{R}^N$. Let f, g be simple functions integrable on E. Let $c \in \mathbb{R}$. Then*

i. $f \pm g$ are integrable on E, with

$$\int_E (f \pm g)(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_E f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \pm \int_E g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}. \tag{9}$$

ii. $c\,f$ is integrable on E, with

$$\int_E (c\,f)(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = c \int_E f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}. \tag{10}$$

iii. $|f|$ is integrable on E.

iv. $f\,g$ is integrable on E.

Remark 8. Note that is usually no clean relation between $\int\ (f g)$ and $(\int\ f)(\int\ g)$.

Proof. Exercise. □

Exercise 3. Let $E \subseteq \mathbb{R}^N$. Let f, g be simple functions integrable on E. Prove that $\max(f, g), \min(f, g)$ are also integrable on E.

Theorem 9. *Let f be a simple function integrable on E_1 and also on E_2. Then f is integrable on $E_1 \cap E_2$, $E_1 \cup E_2$, $E_1 - E_2$. In general we have*

$$\int_{E_1 \cup E_2} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_{E_1} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \int_{E_2} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \int_{E_1 \cap E_2} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}. \tag{11}$$

Thus when $\mu(E_1 \cap E_2) = 0$ we have

$$\int_{E_1 \cup E_2} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_{E_1} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \int_{E_2} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}. \tag{12}$$

Proof. Let $f(\boldsymbol{x}) = \sum_{i=1}^n c_i\,1_{A_i}(\boldsymbol{x})$ with $c_i \neq 0$ for all i, $c_i \neq c_j$, $A_i \cap A_j = \varnothing$ for all $i \neq j$.

- $E_1 \cap E_2$.

  Since f is integrable on E_1 and E_2, by definition $A_i \cap E_1$, $A_i \cap E_2$ are measurable. As a consequence

  $$A_i \cap (E_1 \cap E_2) = (A_i \cap E_1) \cap (A_i \cap E_2) \tag{13}$$

  is measurable. Therefore f is integrable on $E_1 \cap E_2$.

- $E_1 \cup E_2$. Integrability follows from

$$A_i \cap (E_1 \cup E_2) = (A_1 \cap E_1) \cup (A_i \cup E_2). \tag{14}$$

  To prove (11) it suffices to show that if $E_1 \cap E_2 = \varnothing$, then $\int_{E_1 \cup E_2} f(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = \int_{E_1} f(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} + \int_{E_2} f(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$. The details are left as exercise.

- $E_1 - E_2$. Similar to the above.

□

Exercise 4. Let f be a simple function integrable on E_1, E_2. Further assume $E_1 \cap E_2 = \varnothing$. Prove that

$$\int_{E_1 \cup E_2} f(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = \int_{E_1} f(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} + \int_{E_2} f(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}. \tag{15}$$

Exercise 5. Let f be a simple function integrable on E_1, E_2. Prove

$$\int_{E_1} f(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = \int_{E_1 \cap E_2} f(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} + \int_{E_1 - E_2} f(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}. \tag{16}$$