## Jordan measurability

Finally we define Jordan measure for general sets through approximation using simple graphs.

**Definition 1.** **(Jordan inner and outer measure)** *Let* $A \subseteq \mathbb{R}^N$*. Let*

$$W_{\text{in}} := \{B \subseteq \mathbb{R}^N |\ B \text{ is a simple graph and } B \subseteq A\}; \tag{1}$$

$$W_{\text{out}} = \{C \subseteq \mathbb{R}^N |\ C \text{ is a simple graph and } A \subseteq C\}. \tag{2}$$

*We define*

- **(Jordan inner measure)**

$$\mu_{\text{in}}(A) := \sup_{B \in W_{\text{in}}} \mu(B); \tag{3}$$

- **(Jordan outer measure)**

$$\mu_{\text{out}}(A) := \inf_{C \in W_{\text{out}}} \mu(C). \tag{4}$$

**Theorem 2.** *Let*

$$W'_{\text{in}} := \{B \subseteq \mathbb{R}^N |\ B \text{ is a simple graph and } B \subseteq A^o\}; \tag{5}$$

$$W'_{\text{out}} = \{C \subseteq \mathbb{R}^N |\ C \text{ is a simple graph and } \bar{A} \subseteq C\}. \tag{6}$$

*Then*

$$\mu_{\text{in}}(A) = \sup_{B \in W'_{\text{in}}} \mu(B), \qquad \mu_{\text{out}}(A) = \inf_{C \in W'_{\text{out}}} \mu(C). \tag{7}$$

**Exercise 1.** Prove the above theorem.

**Definition 3.** **(Jordan measure)** *Let* $A \subseteq \mathbb{R}^N$*. It is Jordan measurable if and only if* $\mu_{\text{in}}(A) = \mu_{\text{out}}(A)$*. We denote this common value by* $\mu(A)$*.*

**Lemma 4.** *If* $A \subseteq \mathbb{R}^N$ *is Jordan measurable, then* $A$ *is bounded.*

**Remark 5.** Note that $\mu_{\text{in}}(A)$ and $\mu_{\text{out}}(A)$ is defined for all $A \subseteq \mathbb{R}^N$, even those that are not measurable.

The following lemmas are trivial to prove but very useful.

**Lemma 6.** *Let* $A \subseteq \mathbb{R}^N$*. Then* $\mu_{\text{in}}(A) \leqslant \mu_{\text{out}}(A)$*.*

**Lemma 7.** *Let* $A \subseteq \mathbb{R}^N$*. Then*

- *$A$ is measurable if for every $\varepsilon > 0$, there are simple graphs $B, C$ such that $B \subseteq A^o \subseteq \bar{A} \subseteq C$ and $\mu(C) - \mu(B) < \varepsilon$;*

- *$A$ is not measurable if there is $\varepsilon_0 > 0$ such that for all simple graphs $B, C$ satisfying $B \subseteq A^o \subseteq \bar{A} \subseteq C$, there holds $\mu(C) - \mu(B) \geqslant \varepsilon_0$.*

**Exercise 2.** Prove the above lemmas.

**Example 8.** Let $I$ be a compact interval in $\mathbb{R}^N$. Let $A \subseteq \mathbb{R}^N$ be such that $I^o \subseteq A \subseteq I$. Then $A$ is measurable and $\mu(A) = \mu(I)$. In particular $\mu(I^o) = \mu(I)$.

**Proof.** Since $A \subseteq I$ and $I$ is closed, we have $\bar{A} \subseteq I$ and therefore $\mu_{\text{out}}(A) \leqslant \mu(I)$.

On the other hand, take any $a \in (0, 1)$ and any $\boldsymbol{x}_0 \in I^o$, define

$$J := a\,(I - \boldsymbol{x}_0) + \boldsymbol{x}_0, \tag{8}$$

we have $J \subseteq I^o \subseteq A^o$. Consequently $\mu_{\text{in}}(A) \geqslant \mu(J) = a^N \mu(I)$.

By the arbitrariness of $a$ we have $\mu_{\text{in}}(A) \geqslant \mu_{\text{out}}(A)$ so they must be equal. □

**Exercise 3.** Let $A \subseteq \mathbb{R}^N$ be such that there is a simple graph $B$ such that $B^o \subseteq A \subseteq B$. Then $A$ is measurable and $\mu(A) = \mu(B)$.

**Example 9.** Let $A = [0, 1] \cap \mathbb{Q}$. Is $A$ Jordan measurable?

**Solution.** Sincer $\bar{A} = [0, 1]$ we have $\mu_{\text{out}}(A) = 1$. On the other hand $A^o = \varnothing$ so $\mu_{\text{in}}(A) = 0$. Therefore $A$ is not Jordan measurable.

**Exercise 4.** Let $A \subseteq \mathbb{R}^N$ be open. Is $A$ always measurable?

**Example 10.** Let $A \subseteq [0, 1] \times [0, 1]$ be defined as follows:

$$(x, y) \in A \Longleftrightarrow [x \in \mathbb{Q}, y \in [0, 1]] \text{ or } [x \notin \mathbb{Q}, y \in [0, 1/2]]. \tag{9}$$

Then $\bar{A} = [0, 1] \times [0, 1]$ but $A^o = (0, 1) \times (0, 1/2)$ so $A$ is not Jordan measurable. On the other hand, notice that for each fixed $x_0$, $A \cap \{x = x_0\} \subseteq \mathbb{R}$ is Jordan measurable.

The following is a simple criterion checking Jordan measurability.

**Theorem 11.** *A bounded set $A \subseteq \mathbb{R}^N$ is Jordan measurable if and only if $\mu(\partial A) = 0$.*

**Proof.**

- If. Let $\varepsilon > 0$ be arbitrary, we prove $\mu_{\text{out}}(A) \leqslant \mu_{\text{in}}(A) + \varepsilon$. Together with $\mu_{\text{out}}(A) \geqslant \mu_{\text{in}}(A)$, we conclude $\mu_{\text{in}}(A) = \mu_{\text{out}}(A)$.
  - Case 1. $\mu_{\text{in}}(A) = 0$. In this case it must be that $A^o = \varnothing$ since otherwise there is $\boldsymbol{x} \in \mathbb{R}^N, r > 0$ such that $B(\boldsymbol{x}, r) \subseteq A$ and consequently

    $$I := \left[\boldsymbol{x} - \frac{r}{2\sqrt{N}}, \boldsymbol{x} + \frac{r}{2\sqrt{N}}\right]^N \subseteq A \tag{10}$$

    which means

    $$\mu_{\text{in}}(A) \geqslant \left(\frac{r}{\sqrt{N}}\right)^N > 0. \tag{11}$$

Since $A^o = \varnothing$ we have $A \subseteq \partial A$, and $\mu(\partial A) = 0 \Longrightarrow \mu(A) = 0$.

- Case 2. $\mu_{\text{in}}(A) > 0$. In this case let $J_1, \ldots, J_m$ be compact intervals covering $\partial A$ and satisfying

$$\sum_{k=1}^{m} \mu(J_k) < \varepsilon. \tag{12}$$

By "expanding" $J_k$'s a bit we can assume $\partial A \subseteq \cup_{k=1}^m J_k^o$.

Now for each $\boldsymbol{x} \in A^o$, there is an interval $I_{\boldsymbol{x}}$ such that

$$\boldsymbol{x} \in I_{\boldsymbol{x}} \subseteq A^o. \tag{13}$$

As a consequence

$$\bar{A} \subseteq (\cup_{\boldsymbol{x} \in A^o} I_{\boldsymbol{x}}^o) \cup (\cup_{k=1}^m J_m^o). \tag{14}$$

By Heine-Borel $\bar{A}$ is compact. Therefore there are finitely many $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in A^o$ such that

$$\bar{A} \subseteq (\cup_{l=1}^m I_{\boldsymbol{x}_l}) \cup (\cup_{k=1}^m J_m). \tag{15}$$

Consequently

$$\mu_{\text{out}}(A) \leqslant \sum_{l=1}^{n} \mu(I_{\boldsymbol{x}_l}) + \sum_{k=1}^{m} \mu(J_m). \tag{16}$$

On the other hand, consider a "refinement" of $I_{\boldsymbol{x}_l}$ we see that

$$\mu_{\text{in}}(A) \geqslant \sum_{l=1}^{n} \mu(I_{\boldsymbol{x}_l}) \tag{17}$$

and the proof ends.

- Only if.

Assume the contrary: There is $\varepsilon_0 > 0$ such that any $\{J_k\}_{k=1}^m$ covering $\partial A$, $\sum_{k=1}^m \mu(J_k) \geqslant \varepsilon_0$. Now take any $\{I_l\}, \{J_k\}$ such that $I_l^o \cap I_m^o = \varnothing$ and $\cup I_l \subseteq A^o$; $\cup J_k \supseteq \bar{A}$, then through refinement we can write

$$(\cup J_k) - (\cup I_l) \tag{18}$$

into a union of compact intervals. These intervals cover $\partial A$ and consequently

$$\sum \mu(J_k) - \sum \mu(I_l) \geqslant \varepsilon_0 \Longrightarrow \mu_{\text{out}}(A) - \mu_{\text{in}}(A) \geqslant \varepsilon_0 > 0 \tag{19}$$

and $A$ cannot be Jordan measurable. □

**Exercise 5.** Let $A \subseteq \mathbb{R}^N$ be measurable. Then so are $A^o$ and $\bar{A}$ and furthermore $\mu(A^o) = \mu(\bar{A}) = \mu(A)$.

**Example 12.** The unit ball $B := \{\boldsymbol{x} \in \mathbb{R}^N | \ \|\boldsymbol{x}\| < 1\}$ is Jordan measurable.

**Solution.** Clearly $\partial B = \{\boldsymbol{x} \in \mathbb{R}^N | \ \|\boldsymbol{x}\| = 1\}$ which can be written as

$$\partial B = \{\boldsymbol{x} \in \mathbb{R}^N | \ f(\boldsymbol{x}) = 0\} \tag{20}$$

with

$$f(\boldsymbol{x}) = \|\boldsymbol{x}\|^2 - 1. \tag{21}$$

Since grad $f = 2\,\boldsymbol{x} \neq \mathbf{0}$ for every $\boldsymbol{x} \in \partial B$, $\mu(\partial B) = 0$.

**Theorem 13.** *Let $A_1, \ldots, A_n$ be Jordan measurable. Then so are $\cup_{k=1}^n A_k, \cap_{k=1}^n A_k$ and $A_k - A_l$ for any $k, l$.*

**Exercise 6.** Prove that $\mu(\cup_{k=1}^n A_k) \leqslant \sum_{k=1}^n \mu(A_k)$, and equality holds if for every $i \neq j$, $A_i^o \cap A_j^o = \varnothing$.

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**Problem 1.** Let $A := \{(x, y) | \, x \in [a, b], y \in [0, f(x)]\}$ for some single variable function $f(x)$. Then $A$ is Jordan measurable if and only if $f(x)$ is Riemann integrable. Furthermore $\mu(A) = \int_a^b f(x)\,\mathrm{d}x$.

**Problem 2.** Let $A \subseteq \mathbb{R}^M$, $B \subseteq \mathbb{R}^N$ be both Jordan measurable. Then so is $A \times B := \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^{M+N} | \, \boldsymbol{x} \in A, \boldsymbol{y} \in B\}$. Furthermore

$$\mu(A \times B) = \mu(A)\,\mu(B). \tag{22}$$

**Problem 3.** Let $E \subseteq \mathbb{R}^N$ be Jordan measurable. Let $\boldsymbol{l}: \mathbb{R}^N \mapsto \mathbb{R}^N$ be a linear transform. Then $\boldsymbol{l}(E)$ is also Jordan measurable and

$$\mu(\boldsymbol{l}(E)) = |\det A|\,\mu(E) \tag{23}$$

where $A \in \mathbb{R}^{N \times N}$ is the matrix representation of $\boldsymbol{l}$.

**Problem 4.** Let $A \subseteq \mathbb{R}^N$. Then $A$ is Jordan measurable if and only if for every $B \subseteq \mathbb{R}^N$,

$$\mu_{\text{out}}(B \cap A) + \mu_{\text{out}}(B - A) = \mu_{\text{out}}(B). \tag{24}$$

Note that $B$ may or may not be measurable.