Definitions

Second order partial derivatives

Definition 1. *Let $f: \mathbb{R}^N \mapsto \mathbb{R}$. If the j-th partial derivative of $\frac{\partial f}{\partial x_i}: \mathbb{R}^N \mapsto \mathbb{R}$ exists at $\boldsymbol{x}_0 \in \mathbb{R}^N$, then we call $\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)$ a second order partial derivative for the function f at $\boldsymbol{x}_0$.*

Remark 2. Clearly we can define second order partial derivatives for vector functions in a similar manner.

Notation. *Usually we simply denote*

$$\frac{\partial^2 f}{\partial x_j \partial x_i} := \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right). \tag{1}$$

When $j = i$, we write

$$\frac{\partial^2 f}{\partial x_j^2} := \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_i}\right). \tag{2}$$

Example 3. Let $f(x, y) = x \sin y$. Find $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial y^2}$.

Solution. We have

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &:= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(x \sin y)\right) \\ &= \frac{\partial}{\partial x}(\sin y) \\ &= 0. \end{aligned} \tag{3}$$

Similarly

$$\frac{\partial^2 f}{\partial x \partial y} = \cos y; \qquad \frac{\partial^2 f}{\partial y \partial x} = \cos y; \qquad \frac{\partial^2 f}{\partial y^2} = -x \sin y. \tag{4}$$

We observe that $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$. However,

Example 4. Let

$$f(x, y) := \begin{cases} x\, y \dfrac{x^2 - y^2}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}. \tag{5}$$

We calculate $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$.

- $\frac{\partial^2 f}{\partial x \partial y}$.

  First calculate for $(x, y) \neq (0, 0)$,

$$\frac{\partial f}{\partial y} = x \left[\frac{x^2 - y^2}{x^2 + y^2} - \frac{4\, x^2\, y^2}{(x^2 + y^2)^2}\right]. \tag{6}$$

at $(0,0)$ since $f(0,y) \equiv 0$, we have $\frac{\partial f}{\partial y}(0,0) = 0$. Thus

$$\frac{\partial f}{\partial y} = \begin{cases} x\left[\frac{x^2-y^2}{x^2+y^2} - \frac{4\,x^2\,y^2}{(x^2+y^2)^2}\right] & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}. \tag{7}$$

Now we calculate $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ at $(x,y) \neq (0,0)$:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{x^6 + 9\,x^4\,y^2 - 9\,x^2\,y^4 - y^6}{(x^2+y^2)^3}. \tag{8}$$

At $(x,y) = (0,0)$, we have

$$\frac{\partial f}{\partial y}(x,0) = x \Longrightarrow \frac{\partial^2 f}{\partial x \partial y}(0,0) = 1. \tag{9}$$

- $\frac{\partial^2 f}{\partial y \partial x}$. Similarly we have

$$\frac{\partial f}{\partial x} = \begin{cases} y\left[\frac{x^2-y^2}{x^2+y^2} + \frac{4\,x^2\,y^2}{(x^2+y^2)^2}\right] & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}. \tag{10}$$

At $(x,y) \neq (0,0)$,

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{x^6 + 9\,x^4\,y^2 - 9\,x^2\,y^4 - y^6}{(x^2+y^2)^3}. \tag{11}$$

At $(0,0)$,

$$\frac{\partial f}{\partial x}(0,y) = -y \Longrightarrow \frac{\partial^2 f}{\partial y \partial x}(0,0) = -1. \tag{12}$$

Observation.

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \tag{13}$$

when $(x,y) \neq (0,0)$ but they differ at $(0,0)$.

Exercise 1. Prove that $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial x}$ are continuous functions.

Exercise 2. Prove that $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are both continuous everywhere except at $(0,0)$.

Theorem 5. *Let $f(x,y)\colon \mathbb{R}^2 \mapsto \mathbb{R}$. Assume that $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are both continuous at (x_0, y_0), then*

$$\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0). \tag{14}$$

Proof. Applying MVT twice to $A := f(x,y) - f(x_0,y) - f(x,y_0) + f(x_0,y_0)$:
First let $\varphi(y) := f(x,y) - f(x_0,y)$. Then

$$\begin{aligned} A &= \varphi(y) - \varphi(y_0) \\ &= \varphi'(\eta)\,(y - y_0) \\ &= \left[\frac{\partial f}{\partial y}(x,\eta) - \frac{\partial f}{\partial y}(x_0,\eta)\right](y - y_0) \\ &= \frac{\partial^2 f}{\partial x \partial y}(\eta', \eta)\,(y - y_0)(x - x_0). \end{aligned} \tag{15}$$

Similarly, letting $\psi(x) := f(x, y) - f(x, y_0)$ we have

$$A = \psi(x) - \psi(x_0) = \frac{\partial^2 f}{\partial y \partial x}(\xi', \xi)(x - x_0)(y - y_0). \tag{16}$$

Therefore

$$\frac{\partial^2 f}{\partial x \partial y}(\eta', \eta) = \frac{\partial^2 f}{\partial y \partial x}(\xi', \xi). \tag{17}$$

Note that η, η', ξ, ξ' all depend on (x, y) but on the other hand satisfy

$$\|(\xi', \xi) - (x_0, y_0)\|, \|(\eta', \eta) - (x_0, y_0)\| \leqslant \|(x, y) - (x_0, y_0)\|. \tag{18}$$

Letting $(x,\, y) \longrightarrow (x_0,\, y_0)$ and taking advatage of the continuity of $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$, we reach the desired conclusion. □

Exercise 3. What if we directly apply MVT twice to

$$A = [f(x, y) - f(x_0, y)] - [f(x, y_0) - f(x_0, y_0)] \tag{19}$$

without introducing auxiliary functions such as $\varphi(y)$? Can we still prove the theorem?

Exercise 4. Prove that there is no $f: \mathbb{R}^2 \mapsto \mathbb{R}$ such that

$$\frac{\partial f}{\partial x} = y, \qquad \frac{\partial f}{\partial y} = e^x \tag{20}$$

for all $(x, y) \in \mathbb{R}^2$.

Problem 1. (PKUP) Let $f: \mathbb{R}^2 \mapsto \mathbb{R}$. Assume that $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x \partial y}$ are continuous at (x_0, y_0), then

$$\frac{\partial^2 f}{\partial y \partial x} \text{ exists and is continuous at } (x_0, y_0) \text{, and furthermore } \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0) = \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0). \tag{21}$$

Problem 2. (PKUP) Let $f: \mathbb{R}^2 \mapsto \mathbb{R}$. Assume $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are differentiable at (x_0, y_0). Then

$$\frac{\partial^2 f}{\partial y \partial x}(x_0, y_0) = \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0). \tag{22}$$

Higher order partial derivatives

The definition is similar:

$$\frac{\partial^n f}{\partial x_{k_n} \partial x_{k_{n-1}} \cdots \partial x_{k_1}} := \frac{\partial}{\partial x_{k_n}} \left(\frac{\partial^{n-1} f}{\partial x_{k_{n-1}} \cdots \partial x_{k_1}} \right). \tag{23}$$

Example 6. Let $f(x, y) := x^3 y^2$, calculate $\frac{\partial^3 f}{\partial x^2 \partial y}, \frac{\partial^3 f}{\partial x \partial y^2}$.

Solution. We have

$$\begin{aligned} \frac{\partial^3 f}{\partial x^2 \partial y} &:= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y} \right) \right) \\ &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(2\, x^3\, y) \right) \\ &= \frac{\partial}{\partial x}(6\, x^2\, y) \\ &= 12\, x\, y. \end{aligned} \tag{24}$$

and similarly

$$\frac{\partial^3 f}{\partial x \partial y^2} = 6\,x^2. \tag{25}$$

Exercise 5. (PKUP) Let $f(x, y) = x^3 \sin y + y^3 \sin x$. Find $\frac{\partial^6 f}{\partial x^3 \partial y^3}$.

Exercise 6. (PKUP) Let $f(x, y) = \sin(x^2 + y^2)$. Find $\frac{\partial^3 f}{\partial x^2}$.

Exercise 7. (PKUP) Let

$$f(x, y) := \begin{cases} \exp[-1/(x^2 + y^2)] & x^2 + y^2 \neq 0 \\ 0 & x^2 + y^2 = 0 \end{cases}. \tag{26}$$

Find $\frac{\partial^2 f}{\partial x^2}(0, 0)$, $\frac{\partial^2 f}{\partial x \partial y}(0, 0)$.

Exercise 8. (PKUP) Prove that

$$u(x, t) := \frac{1}{2\,a\,\sqrt{\pi\,t}} \exp\left[-\frac{(x - b)^2}{4\,a^2\,t}\right] \tag{27}$$

where $a, b \in \mathbb{R}$, satisfies

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}. \tag{28}$$

Exercise 9. (PKUP) Let $f(x, y) := (x - x_0)^p\,(y - y_0)^q$ with $p, q \in \mathbb{N} \cup \{0\}$. Find $\frac{\partial^{p+q} u}{\partial x^p\,\partial y^q}$.

Exercise 10. (PKUP) Let $f(x, y) := \frac{x + y}{x - y}$. Find $\frac{\partial^{m+n} f}{\partial x^m \partial y^n}$ where $m, n \in \mathbb{N} \cup \{0\}$, $x \neq y$.

Exercise 11. (PKUP) Let $f(x, y) := \ln(a\,x + b\,y)$. Find $\frac{\partial^{m+n} f}{\partial x^m \partial y^n}$.

Exercise 12. (PKUP) Let $f(x, y, z) := x\,y\,z\,e^{x+y+z}$. Find $\frac{\partial^{p+q+r} f}{\partial x^p \partial y^q \partial z^r}$ where $p, q, r \in \mathbb{N} \cup \{0\}$.

Problem 3. State and prove the theorem about order of taking derivatives for higher order partial derivatives of $f : \mathbb{R}^N \mapsto \mathbb{R}$.

Problem 4. Solve the equation (assume all second order partial derivatives of u are continuous.)

$$3\,\frac{\partial^2 u}{\partial x^2} - 4\,\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{29}$$

through transforming it to $\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$ via the change of variable $\xi = a\,x + b\,y$, $\eta = c\,x + d\,y$ for some appropriate constants a, b, c, d.

Problem 5. Prove that under the change of variables $x = r\cos\theta$, $y = r\sin\theta$, the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{30}$$

is transformed to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0. \tag{31}$$