## Implicit function theorem in $\mathbb{R}^2$

We now consider the equation

$$f(x,y)=0 \tag{1}$$

where $f:\mathbb{R}^2\mapsto\mathbb{R}$ and try to solve it near $(x_0,y_0)$.

**Theorem 1. (Implicit function theorem in $\mathbb{R}^2$)** *Let $f:\mathbb{R}^2\mapsto\mathbb{R}$ satisfy*

*i.* *$f(x,y)$ is partially differentiable with continuous partial derivatives;*

*ii.* *$f(x_0,y_0)=0$;*

*iii.* *$\frac{\partial f}{\partial y}(x_0,y_0)\neq 0$.*

*Then there is an open interval $I\times J$ such that $(x_0,y_0)\in I\times J$ and*

*i.* *For every $x\in I$ there is a unique $y\in J$ such that $f(x,y)=0$. Thus we can define the implicit function $Y(x):=y$.*

*ii.* *$Y(x_0)=y_0$;*

*iii.* *$Y$ is differentiable with continuous derivatives;*

*iv.* *For $x\in I$,*

$$Y'(x)=-\left(\frac{\partial f}{\partial x}(x,Y(x))\right)\Big/\left(\frac{\partial f}{\partial y}(x,Y(x))\right). \tag{2}$$

**Proof.** To simplify presentation, let $a:=\frac{\partial f}{\partial y}(x_0,y_0)$. We choose $I\times J:=(x_0-\delta_1,x_0+\delta_1)\times(y_0-\delta_2,y_0+\delta_2)$ with $\delta_1,\delta_2$ satisfying the following.

1. $\delta_2$ small enough so that $\frac{\partial f}{\partial y}(x,y)\in\left(\frac{a}{2},\frac{3a}{2}\right)$ for all $\|(x,y)-(x_0,y_0)\|<2\,\delta_2$. This is possible as $a\neq 0$, and $\frac{\partial f}{\partial y}$ is continuous.
2. For the above $\delta_2$, we further choose $\delta_1\leqslant\delta_2$ such that for all $x\in(x_0-\delta_1,x_0+\delta_1)$, $\left|\frac{2f(x,y_0)}{a}\right|<\delta_2$.

Combining these two, we see that for all $(x,y)\in I\times J$,

$$\frac{\partial f}{\partial y}(x,y)\in\left(\frac{a}{2},\frac{3a}{2}\right) \text{ and } \left|\frac{2f(x,y_0)}{a}\right|<\delta_2 \tag{3}$$

- First we try to define the function $Y$ for $x\in I$. That is for each $x$, we try to find $y$ solving $g(y)=0$ where

$$g(y):=F(x,y). \tag{4}$$

  Start from $y_0$, we define for every $n\geqslant 1$,

$$y_n=y_{n-1}-\frac{g(y_{n-1})}{a}. \tag{5}$$

We prove the following:

For all $n \geqslant 1$, $|y_n - y_0| < \delta_2$, and $|y_{n+1} - y_n| \leqslant \frac{|y_n - y_{n-1}|}{2}$.

We prove through induction.

- Base step. We have

$$y_1 = y_0 - \frac{g(y_0)}{a} \Longrightarrow |y_1 - y_0| = \frac{|f(x, y_0)|}{a} < \frac{\delta_2}{2} \tag{6}$$

Now consider

$$y_2 = y_1 - \frac{g(y_1)}{a} \tag{7}$$

and we have

$$y_2 - y_1 = y_1 - y_0 - \frac{g(y_1) - g(y_0)}{a} = \left(1 - \frac{g'(\xi)}{a}\right)(y_1 - y_0) \tag{8}$$

for some $\xi \in (y_0, y_1)$.

Since both $y_1, y_0 \in J$, we have $g'(\xi) \in \left(\frac{a}{2}, \frac{3\,a}{2}\right)$ and consequently

$$|y_2 - y_1| < \frac{|y_1 - y_0|}{2}. \tag{9}$$

- Induction step. Assume the claim holds for all $n = 1, \ldots, k-1$. Now consider the case $n = k$. First by induction assumption

$$|y_k - y_{k-1}| \leqslant \frac{|y_{k-1} - y_{k-2}|}{2} \leqslant \frac{|y_{k-2} - y_{k-3}|}{2^2} \leqslant \cdots \leqslant \frac{|y_1 - y_0|}{2^{k-1}}, \tag{10}$$

We have

$$|y_k - y_0| < 2\,|y_1 - y_0| < \delta_2. \tag{11}$$

In particular, $y_k, y_{k-1} \in J$.

Now we have

$$\begin{aligned} (y_{k+1} - y_k) &= (y_k - y_{k-1}) - \frac{g(y_k) - g(y_{k-1})}{a} \\ &= \left(1 - \frac{g'(\xi)}{a}\right)(y_k - y_{k-1}). \end{aligned} \tag{12}$$

Since $y_k, y_{k-1} \in J$, we have $\xi \in J$. By our choice of $I$ and $J$, we have $g'(\xi) = \frac{\partial F(x, \xi)}{\partial y} \in \left(\frac{a}{2}, \frac{3\,a}{2}\right)$ which gives

$$|y_{k+1} - y_k| \leqslant \frac{|y_k - y_{k-1}|}{2}. \tag{13}$$

Now we can conclude that $\{y_n\}$ is a Cauchy sequence and therefore converges: $\lim_{n \longrightarrow \infty} y_n = y \in J$. Now taking limit $n \longrightarrow \infty$ in both sides of (5) we have

$$y = y - \frac{g(y)}{a} \Longrightarrow g(y) = 0 \text{ that is } F(x, y) = 0. \tag{14}$$

Note that here we have used the continuity of $g(y)$ which is a consequence of the continuity of $f(x,y)$ which is in turn a consequence of the differentiability of $f$ which in turn follows from the assumption that $f$'s partial derivatives are continuous.

- Step 2. We prove that $y$ is unique in $J$. That is, if $f(x,y_1)=f(x,y_2)=0$ for $y_1,y_2\in J$, then $y_1=y_2$. For such $y_1,y_2$ we would have some $\xi\in(y_1,y_2)$ such that

$$0=f(x,y_1)-f(x,y_2)=\frac{\partial f(x,\xi)}{\partial y}(y_1-y_2). \tag{15}$$

Now we know that for all $(x,y)\in I\times J$, $\frac{\partial f(x,y)}{\partial y}\in\left(\frac{a}{2},\frac{3a}{2}\right)$ for some $a\neq 0$. Therefore $\frac{\partial f(x,\xi)}{\partial y}\neq 0$ which means $y_1-y_2=0$.

- Step 3. We prove differentiability of $Y$ and calculate the differential. Since

$$F(x,Y(x))=0 \tag{16}$$

we have

$$F(x+\delta x,Y(x+\delta x))=0. \tag{17}$$

By mean value theorem we have

$$F(x+\delta x,Y(x+\delta x))-F(x,Y(x+\delta x))+F(x,Y(x+\delta x))-F(x,y)=0 \tag{18}$$

which gives

$$\frac{\partial F}{\partial x}(\xi)\,\delta x+\frac{\partial F}{\partial y}(\xi)\,(Y(x+\delta x)-Y(x))=0 \tag{19}$$

which gives

$$\frac{Y(x+\delta x)-Y(x)}{\delta x}=-\frac{F_x}{F_y}. \tag{20}$$

This proves differentiability together with the continuity of the derivative. □

**Problem 1.** The above proof still works for $f:\mathbb{R}^N\mapsto\mathbb{R}$. Figure out the details.

**Theorem 2. (Inverse function theorem)** *Let $f:\mathbb{R}\mapsto\mathbb{R}$ be differentiable with continuous derivative. Let $y_0\in\mathbb{R}$ and set $x_0=f(y_0)$. Then if $f'(y_0)\neq 0$, there are intervals $I\ni x_0, J\ni y_0$ such that there is a function $g$ satisfying $f(g(x))=x$ fo all $x\in J$.*

**Exercise 1.** Prove the above theorem.

**Example 3.** Let $y=Y(x)$ be defined through $Y(1)=1$ and

$$x^2\,y^2-3\,y+2\,x^3=0. \tag{21}$$

Find $Y'(1)$.

**Solution.** First check

$$\frac{\partial(x^2\,y^2-3\,y+2\,x^3)}{\partial y}(1,1)=(2\,x^2\,y-3)|_{x=1,y=1}=-1\neq 0 \tag{22}$$

So the implicit function exists. Now taking $\frac{\mathrm{d}}{\mathrm{d}x}$ to

$$x^2\, Y(x)^2 - 3\, Y(x) + 2\, x^3 = 0 \tag{23}$$

we have

$$2\, x\, Y(x)^2 + 2\, x^2\, Y(x)\, Y'(x) - 3\, Y'(x) + 6\, x^2 = 0. \tag{24}$$

Setting $x = 1$ we have

$$2 + 2\, Y'(1) - 3\, Y'(1) + 6 = 0 \Longrightarrow Y'(1) = 8. \tag{25}$$

**Remark 4.** As can be seen in the above example, often it is simpler to use chain rule instead of trying to remember the formula $Y'(x) = -\left(\frac{\partial f}{\partial x}(x, Y(x))\right) / \left(\frac{\partial f}{\partial y}(x, Y(x))\right)$.

**Example 5.** Let $z = Z(x, y)$ be defined through

$$\sin z - x\, y\, z = 0. \tag{26}$$

Find $\frac{\partial Z}{\partial x}, \frac{\partial Z}{\partial y}$.

**Solution.** First we check

$$\frac{\partial(\sin z - x\, y\, z)}{\partial z} = \cos z - x\, y. \tag{27}$$

So the theorem can be applied at points where $\cos z - x\, y \neq 0$. Then we can easily obtain

$$\frac{\partial z}{\partial x} = \frac{y\, z}{\cos z - x\, y}, \qquad \frac{\partial z}{\partial y} = \frac{x\, z}{\cos z - x\, y}. \tag{28}$$

**Exercise 2.** Let $F$ be continuously differentiable. Consider $F(x, y, z) = 0$. Prove $\frac{\partial x}{\partial y} \frac{\partial y}{\partial z} \frac{\partial z}{\partial x} = -1$.