Motivation

Consider the equation for the unit circle S:

$$x^2 + y^2 = 1. \tag{1}$$

We see that if we consider $y > 0$, we can write y as a function of x:

$$y = \sqrt{1 - x^2}. \tag{2}$$

Similarly, for $y < 0$, we can write

$$y = -\sqrt{1 - x^2}. \tag{3}$$

Summarizing, given any $(x_0, y_0) \in S$ with $y_0 \neq 0$, then there is an open set E containing x_0 such that there is $f: E \mapsto \mathbb{R}$ such that $y = f(x)$ for $x \in E$. This function is "hidden" in the relation $x^2 + y^2 = 1$. We can say that y is a function of x given "implicitly", or an "implicit function".

Exercise 1. For what (x_0, y_0) can x be written as a function of y?

Similarly, for a function equation $f(x, y, z) = 0$, we expect to be able to "solve" $z = Z(x, y)$ or $x = X(y, z)$ or $y = Y(x, z)$. In other words, $f(x, y, z) = 0$ should be the equation of a surface in $\mathbb{R}^3$.

Example 1. Consider $3\,x + 2\,y - 4z + 7 = 0$. We see that it is possible to re-write it as

$$z = Z(x, y) := \frac{3}{4}\,x + \frac{1}{2}\,y + \frac{7}{4} \tag{4}$$

which is our familiar equation for a plane in $\mathbb{R}^3$.

In general, if we have M equations:

$$\begin{aligned} f_1(x_1, \ldots, x_N) &= 0 \qquad (5) \\ \vdots \quad &\vdots \quad \vdots \\ f_M(x_1, \ldots, x_N) &= 0 \qquad (6) \end{aligned}$$

We are interested in the situation $N > M$, and expect to write M of the x_i's as functions of the other $N - M$ x_i's.

We start from two simpler situations:

1. f_i's are all linear;
2. $f: \mathbb{R}^2 \mapsto \mathbb{R}$.

In the first situation we figure out how to deal with high dimensions, and in the second we figure out how to deal with nonlinearity. Then we will put things together and prove the general situation.

Implicit function theorems for linear functions

We consider the case when $\boldsymbol{f}: \mathbb{R}^N \mapsto \mathbb{R}^M$ is linear. In this case the M equations

$$\begin{aligned} f_1(x_1, \ldots, x_N) &= 0 \qquad (7) \\ \vdots \quad &\vdots \quad \vdots \\ f_M(x_1, \ldots, x_N) &= 0 \qquad (8) \end{aligned}$$

reduces to M linear equations

$$\begin{aligned} a_{11} x_1 + \cdots + a_{1N} x_N &= 0 \quad (9)\\ \vdots \quad & \vdots \quad \vdots \\ a_{M1} x_1 + \cdots + a_{MN} x_N &= 0 \quad (10)\end{aligned}$$

The problem now reduces to solving a system of linear equations.

Example 2. Consider the following system

$$\begin{aligned} 3\, x_1 + x_2 + x_3 &= 0 \quad (11)\\ 2\, x_1 + x_2 + x_3 &= 0 \quad (12)\end{aligned}$$

We check

- Can we write x_1, x_2 as a function of x_3? Solving

  $$\begin{aligned} 3\, x_1 + x_2 &= -x_3 \quad (13)\\ 2\, x_1 + x_2 &= -x_3 \quad (14)\end{aligned}$$

  we have

  $$x_1 = 0, \qquad x_2 = -x_3. \quad (15)$$

- Can we write x_2, x_3 as a function of x_1? Solving

  $$\begin{aligned} x_2 + x_3 &= -3\, x_1 \quad (16)\\ x_2 + x_3 &= -2\, x_1 \quad (17)\end{aligned}$$

  we see that $x_2 = x_3 = x_1 = 0$ which means we cannot write x_2, x_3 as functions of x_1.

- Can we write x_1, x_3 as a function of x_2? Solving

  $$\begin{aligned} 3\, x_1 + x_3 &= -x_2 \quad (18)\\ 2\, x_1 + x_3 &= -x_2 \quad (19)\end{aligned}$$

  we have

  $$x_1 = 0, \qquad x_3 = -x_2. \quad (20)$$

Now if we use the language of matrices, the situation is quite clear: For example, when trying to solve x_1, x_2 using x_3, we write the equation as

$$\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_3 \\ -x_3 \end{pmatrix} \quad (21)$$

and it can be solved if the matrix $\left(\begin{smallmatrix} 3 & 1 \\ 2 & 1 \end{smallmatrix}\right)$ is invertible, or equivalently $\det \left(\begin{smallmatrix} 3 & 1 \\ 2 & 1 \end{smallmatrix}\right) \neq 0$. In this case we have

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -x_3 \\ -x_3 \end{pmatrix}. \quad (22)$$

Note that this "implicit function" is also linear.

We see that it may or may not be possible to write M of the x_i's as functions of the other $N-M$ x_i's. In the following we obtain a sufficient condition.

Theorem 3. *Let $N > M$. Let $\boldsymbol{f}\colon \mathbb{R}^N \mapsto \mathbb{R}^M$ be linear. Then a sufficient and necessary condition for the solution to*

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0} \tag{23}$$

to take the form

$$\begin{pmatrix} x_1 \\ \vdots \\ x_M \end{pmatrix} = \begin{pmatrix} X_1(x_{M+1}, \ldots, x_N) \\ \vdots \\ X_N(x_{M+1}, \ldots, x_N) \end{pmatrix} \tag{24}$$

is

$$\det \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_M} \\ \vdots & & \vdots \\ \frac{\partial f_M}{\partial x_1} & \cdots & \frac{\partial f_M}{\partial x_M} \end{pmatrix} \neq 0. \tag{25}$$

Remark 4. It is clear that we can replace $x_1, \ldots, x_N$ by any N of the variables.

Proof. Let A be the matrix representation of $\boldsymbol{f}$. Then

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_N} \\ \vdots & & \vdots \\ \frac{\partial f_N}{\partial x_1} & \cdots & \frac{\partial f_N}{\partial x_N} \end{pmatrix} = \begin{pmatrix} A_1 & A_2 \end{pmatrix} \tag{26}$$

where

$$A_1 = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_M} \\ \vdots & & \vdots \\ \frac{\partial f_M}{\partial x_1} & \cdots & \frac{\partial f_M}{\partial x_M} \end{pmatrix}, \qquad A_2 = \begin{pmatrix} \frac{\partial f_1}{\partial x_{M+1}} & \cdots & \frac{\partial f_1}{\partial x_N} \\ \vdots & & \vdots \\ \frac{\partial f_M}{\partial x_{M+1}} & \cdots & \frac{\partial f_M}{\partial x_N} \end{pmatrix} \tag{27}$$

and we can write the equation $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$ as

$$A_1 \begin{pmatrix} x_1 \\ \vdots \\ x_M \end{pmatrix} = -A_2 \begin{pmatrix} x_{M+1} \\ \vdots \\ x_N \end{pmatrix}. \tag{28}$$

Now since $\det A_1 \neq 0$, it is invertible and

$$\begin{pmatrix} x_1 \\ \vdots \\ x_M \end{pmatrix} = A_1^{-1} A_2 \begin{pmatrix} x_{M+1} \\ \vdots \\ x_N \end{pmatrix}. \tag{29}$$

Thus ends the proof. □

Remark 5. Note that we can understand the above result from a function point of view: The implicit function $\boldsymbol{g}$ has matrix representation $A_1^{-1} A_2$. The matrices A_1, A_2 are the Jacobian matrices of $\boldsymbol{f}$ with respect to $(x_1, ..., x_M)$ and $(x_{M+1}, ..., x_N)$ respectively. If we denote $\boldsymbol{y} := \begin{pmatrix} x_1 \\ \vdots \\ x_M \end{pmatrix}$ and $\boldsymbol{x} := \begin{pmatrix} x_{M+1} \\ \vdots \\ x_N \end{pmatrix}$, we can write the above results as:

$$D\boldsymbol{g}(\boldsymbol{x}_0) = D_{\boldsymbol{y}}\boldsymbol{f}(\boldsymbol{x}_0, \boldsymbol{y}_0)^{-1} \circ D_{\boldsymbol{x}}\boldsymbol{f}(\boldsymbol{x}_0, \boldsymbol{y}_0). \tag{30}$$

We will see that this is the version that can be generalized to the nonlinear case.

Theorem 6. (Inverse function theorem) *Let $\boldsymbol{f}: \mathbb{R}^N \mapsto \mathbb{R}^N$ be linear. Then $\boldsymbol{f}$ has an inverse function if and only if the determinant of the Jacobian is nonzero.*

Proof. We are solving

$$\boldsymbol{f}(\boldsymbol{y}) = \boldsymbol{x}. \tag{31}$$

Now let $N = 2\,M$and identify $\boldsymbol{y} = \begin{pmatrix} x_1 \\ \vdots \\ x_M \end{pmatrix}, \boldsymbol{x} = \begin{pmatrix} x_{M+1} \\ \vdots \\ x_N \end{pmatrix}$, we can apply the implicit function theorem and obtain the result. □