Convergence of sequences, limit of functions, continuity

With the definition of norm, or more precisely the distance between any two vectors in $\mathbb{R}^N$:

$$\operatorname{dist}(\boldsymbol{x},\boldsymbol{y}) := \|\boldsymbol{x}-\boldsymbol{y}\| := [(x_1-y_1)^2+\cdots+(x_N-y_N)^2]^{1/2}, \tag{1}$$

generalizing convergence of sequences, limit of functions, and continuity to $\mathbb{R}^N$ is almost trivial. All we need to do is to replace the one-dimensional distance, defined through absolute value, by the N-dimensional distance as defined above.

Definition 1. *Let $\{\boldsymbol{x}_n\}$ be a sequence of points in $\mathbb{R}^N$. A point $\boldsymbol{x}_0$ is said to be the limit of the sequence if and only if*

$$\forall \varepsilon>0 \quad \exists K\in\mathbb{N} \quad \forall n>K \qquad \|\boldsymbol{x}_n-\boldsymbol{x}_0\|<\varepsilon. \tag{2}$$

Exercise 1. Prove that this is equivalent to $\lim_{n\longrightarrow\infty}\|\boldsymbol{x}_n-\boldsymbol{x}_0\|=0$. Note that $\{\|\boldsymbol{x}_n-\boldsymbol{x}_0\|\}$ is a sequence of real numbers.

Exercise 2. Prove that this is equivalent to $\lim_{n\longrightarrow\infty}x_n^k=x_0^k$ for $k=1, 2, ..., N$. Here x_n^k denotes the k-th coordinate of the vector $\boldsymbol{x}_n$.

The limit of functions is a bit more complicated.

Definition 2. *Let $\boldsymbol{f}: E\subseteq\mathbb{R}^N\mapsto\mathbb{R}^M$. Let $\boldsymbol{x}_0\in\mathbb{R}^N$ be such that for any $r>0$, $(B(\boldsymbol{x}_0,r)-\{\boldsymbol{x}_0\})\cap E\neq\varnothing$. Then we say $\boldsymbol{y}_0\in\mathbb{R}^M$ is the limit of $\boldsymbol{f}$ at $\boldsymbol{x}_0$, denoted $\boldsymbol{y}_0=\lim_{\boldsymbol{x}\to\boldsymbol{x}_0}\boldsymbol{f}(\boldsymbol{x})$, if and only if*

$$\forall\varepsilon>0 \quad \exists\delta>0 \quad \forall\boldsymbol{x}\in E \text{ satisfying } 0<\|\boldsymbol{x}-\boldsymbol{x}_0\|<\delta \qquad \|\boldsymbol{f}(\boldsymbol{x})-\boldsymbol{y}_0\|<\varepsilon. \tag{3}$$

Exercise 3. Let $\boldsymbol{f}(\boldsymbol{x})$ be defined on $E{:} =\{(x, y)\in\mathbb{R}^2 |\ x>0,\ y>0\}$. For what (x_0, y_0) can the existence of $\lim_{(x,y)\longrightarrow(x_0,y_0)}\boldsymbol{f}(x,y)$ be discussed? Justify your answer.

Example 3. Let $f(x, y, z) := \frac{x\,y\,z}{x^2+y^2+z^2}$ be define for $(x, y, z)\neq(0, 0, 0)$. We prove that $\lim_{(x,y,z)\to(0,0,0)}=0$.

Proof. For any $\varepsilon>0$, we find $\delta>0$ such that $|f(x, y, z)|<\varepsilon$ whenever $\|(x, y, z)\| := (x^2+y^2+z^2)^{1/2}<\delta$. Observe

$$|f(x,y,z)| = \left|\frac{x\,y\,z}{x^2+y^2+z^2}\right| \leqslant \left|\frac{x\,y}{x^2+y^2}\right| |z| \leqslant \frac{1}{2}\,(x^2+y^2+z^2)^{1/2}. \tag{4}$$

Now clearly $\delta=2\,\varepsilon$ does the job. □

Exercise 4. Define $f(x,y) := \frac{x\,y}{x^2+y^2}$ for $(x,y)\neq(0,0)$. Study the limiting behavior of $f(x,y)$ as $(x,y)\longrightarrow(0,0)$.

From definition of limit naturally follows that of continuity.

Definition 4. *Let $\boldsymbol{f}: E\subseteq\mathbb{R}^N\mapsto\mathbb{R}^M$. Let $\boldsymbol{x}_0\in E$ be such that for any $r>0$, $(B(\boldsymbol{x}_0,r)-\{\boldsymbol{x}_0\})\cap E\neq\varnothing$. Then we say $\boldsymbol{f}$ is continuous at $\boldsymbol{x}_0$ if and only if $\lim_{\boldsymbol{x}\to\boldsymbol{x}_0}\boldsymbol{f}(\boldsymbol{x})=\boldsymbol{f}(\boldsymbol{x}_0)$.*

Example 5. Let $f(x,y,z)=\begin{cases}(x+y+z)\sin\frac{1}{x} & x\neq0\\ 0 & x=0\end{cases}$. We prove that f is continuous at $(0,0,0)$.

Proof. For any $\varepsilon > 0$, take $\delta = \varepsilon / 3$. Then for any $\|(x, y, z)\| < \delta$, we have either $f(x, y, z) = 0$ or

$$|f(x, y, z)| \leqslant |x + y + z| < 3\,(x^2 + y^2 + z^2)^{1/2} = \varepsilon. \tag{5}$$

Thus ends the proof. □

Exercise 5. Show that, in the above proof, we can take $\delta = a\,\varepsilon$ for any $a < \sqrt{3}$. (Hint: Cauchy-Schwarz)

Example 6. Let $E := \{0\} \cup \{1/n|\, n \in \mathbb{N}\}$. Define $f(x) := \begin{cases} \frac{n}{n+1} & x = \frac{1}{n},\ n = 1, 2, 3, \ldots \\ 1 & x = 0 \end{cases}$. We prove that f is a continuous function on E.

Proof. One can check that we only need to discuss continuity at $x = 0$. For any $\varepsilon > 0$, take $\delta = \varepsilon$. Then for any $x \in E \cap (-\delta, \delta)$ we have $x = \frac{1}{n}$ for some $n > \delta^{-1} = \varepsilon^{-1}$. For such x we have

$$|f(x) - f(0)| = \frac{1}{n} < \varepsilon \tag{6}$$

which ends the proof. □

Exercise 6. Let $\boldsymbol{f}: E \subseteq \mathbb{R}^N \mapsto \mathbb{R}^M$. Let $\boldsymbol{x}_0 \in \mathbb{R}^N$ be such that for any $r > 0$, $(B(\boldsymbol{x}_0, r) - \{\boldsymbol{x}_0\}) \cap E \neq \varnothing$. Denote the M components of $\boldsymbol{f}$ by $f_1, \ldots, f_M$. Then $\boldsymbol{f}$ is continuous at $\boldsymbol{x}_0$ if and only if $f_1, \ldots, f_M$ are all continuous at $\boldsymbol{x}_0$.

Exercise 7. Define

$$f(x, y) := \begin{cases} \frac{x^3\, y}{x^6 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}. \tag{7}$$

Is $f(x, y)$ continuous at $(0, 0)$? Justify your answer.

Problem 1. In this problem we develop all the major properties of sequence limit.

a) State and prove convergence results for sum, difference, product of sequences;

b) Prove that every sequence in $\mathbb{R}^N$ has at most one limit;

c) Prove that every convergent sequence is bounded;

d) State and prove Cauchy criterion for sequences in $\mathbb{R}^N$;

Problem 2. Let $\boldsymbol{f}: E \mapsto \mathbb{R}^M$. Let $\boldsymbol{x}_0 \in \mathbb{R}^N$ be such that $\lim_{\boldsymbol{x} \to \boldsymbol{x}_0} \boldsymbol{f}(\boldsymbol{x})$ can be discussed. Prove that $\lim_{\boldsymbol{x} \to x_0} \boldsymbol{f}(\boldsymbol{x})$ exists if and only if for every sequence $\{\boldsymbol{x}_n\} \subseteq E$ satisfying $\boldsymbol{x}_0 \notin \{\boldsymbol{x}_n\}$ and $\lim_{n \longrightarrow \infty} \boldsymbol{x}_n = \boldsymbol{x}_0$, the limit $\lim_{n \longrightarrow \infty} \boldsymbol{f}(\boldsymbol{x}_n)$ exists.

Problem 3. In this problem we develop major properties of continuous functions.

a) Let $\boldsymbol{f}, \boldsymbol{g}: E \subseteq \mathbb{R}^N \mapsto \mathbb{R}^M$ be conintuous at $\boldsymbol{x}_0$. Let $\phi: E \mapsto \mathbb{R}$ be continuous at $\boldsymbol{x}_0$. Prove that the following functions are also continuous at $\boldsymbol{x}_0$: $\boldsymbol{f} \pm \boldsymbol{g}$, $\phi\, \boldsymbol{f}$, $\boldsymbol{f} \cdot \boldsymbol{g}$.

b) Let $\boldsymbol{h}: \mathbb{R}^M \mapsto \mathbb{R}^K$ be continuous at $\boldsymbol{f}(\boldsymbol{x}_0)$. Prove that the composite function $\boldsymbol{h} \circ \boldsymbol{f}$ is also continuous at $\boldsymbol{x}_0$.

c) Let $\boldsymbol{f}: E \subseteq \mathbb{R}^N \mapsto F \subseteq \mathbb{R}^M$ be invertible with inverse function $\boldsymbol{g}: F \mapsto E$. If $\boldsymbol{f}$ is continuous at $\boldsymbol{x}_0$, can we conclude that $\boldsymbol{g}$ is continuous at $\boldsymbol{y}_0 := \boldsymbol{f}(\boldsymbol{x}_0)$? Justify your answer.

Open and closed sets

Finer properties of continuous functions in single variable calculus, such as

$f\colon [a,b] \mapsto \mathbb{R}$ reaches its maximum and minimum,

is nontrival to generalize, and requires deeper understanding of topology of $\mathbb{R}^N$.

Exercise 8. Would you agree that $f\colon I \mapsto \mathbb{R}$ reaches its maximum and minimum if I is a closed N-dimensional interval and f is continuous? If you do, can you prove it with what we have so far?

Definitions

Definition 7. (Open/closed sets) *A set $E \subseteq \mathbb{R}^N$ is* open *if and only if, for any $\boldsymbol{x} \in E$, there is $r > 0$ such that the open ball $B(\boldsymbol{x}, r) \subseteq E$.*

A set $E \subseteq \mathbb{R}^N$ is closed *if and only if E^c is open.*

Remark 8. The empty set, $\varnothing$, is open.[1]

Exercise 9. Prove that $\mathbb{R}^N$ is open.

Exercise 10. Let $\boldsymbol{x}_0 \in \mathbb{R}^N$. Prove that the set $\{\boldsymbol{x}_0\}$ is closed.

Example 9. Let $\boldsymbol{x}_0 \in \mathbb{R}^N, r > 0$. Then the open ball $B(\boldsymbol{x}_0, r)$ is open while the closed ball $\overline{B(\boldsymbol{x}_0, r)}$ is closed.

Proof.

- Open ball is open.

  Let $\boldsymbol{x} \in B(\boldsymbol{x}_0, r)$. Then by definition $r_1 := \|\boldsymbol{x} - \boldsymbol{x}_0\| < r$. We claim that $B(\boldsymbol{x}, r - r_1) \subseteq B(\boldsymbol{x}_0, r)$. To see this, take any $\boldsymbol{y} \in B(\boldsymbol{x}, r - r_1)$ and check

  $$\|\boldsymbol{y} - \boldsymbol{x}_0\| \leqslant \|\boldsymbol{y} - \boldsymbol{x}\| + \|\boldsymbol{x} - \boldsymbol{x}_0\| < r - r_1 + r_1 = r. \tag{9}$$

- Closed ball is closed.

  All we need to show is its complement:

  $$E := (\overline{B(\boldsymbol{x}_0, r)})^c = \{\boldsymbol{x} \in \mathbb{R}^N | \, \|\boldsymbol{x} - \boldsymbol{x}_0\| > r\} \tag{10}$$

  is open. Take any $\boldsymbol{x} \in E$ and denote $r_1 := \|\boldsymbol{x} - \boldsymbol{x}_0\| > r$. We show that $B(\boldsymbol{x}, r_1 - r) \subseteq E$. To see this, take any $\boldsymbol{y} \in B(\boldsymbol{x}, r_1 - r)$, we check

  $$\|\boldsymbol{y} - \boldsymbol{x}_0\| \geqslant \|\boldsymbol{x} - \boldsymbol{x}_0\| - \|\boldsymbol{y} - \boldsymbol{x}\| > r_1 - (r_1 - r) = r. \tag{11}$$

  Thus ends the proof. □

Exercise 11. Prove that the sphere $S(\boldsymbol{x}_0, r) := \{\boldsymbol{x} \in \mathbb{R}^N | \, \|\boldsymbol{x} - \boldsymbol{x}_0\| = r\}$ is closed.

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1. This is reasonable. The definition of open sets can be written as

$$(\boldsymbol{x} \in E) \Longrightarrow (\exists r > 0 \quad B(\boldsymbol{x}, r) \subseteq E). \tag{8}$$

When $E = \varnothing$ $\boldsymbol{x} \in E$ is always false, which means the whole statement is true.

Example 10. Let $P \subset \mathbb{R}^N$ be a hyperplane. Then P is closed.

Proof. We need to show that P^c is open. Let the equation for P be $\boldsymbol{a} \cdot \boldsymbol{x} = b$. Then

$$P^c = \{\boldsymbol{x} \in \mathbb{R}^N | \, \boldsymbol{a} \cdot \boldsymbol{x} \neq b\}. \tag{12}$$

Take any $\boldsymbol{x} \in P^c$. Take any $\boldsymbol{x}_0 \in P$. Then we have

$$r := \frac{|\boldsymbol{a} \cdot (\boldsymbol{x} - \boldsymbol{x}_0)|}{\|\boldsymbol{a}\|} > 0. \tag{13}$$

We claim that $B(\boldsymbol{x}, r) \subset P^c$. Let $\boldsymbol{y} \in B(\boldsymbol{x}, r)$. We check

$$\begin{aligned} |\boldsymbol{a} \cdot (\boldsymbol{y} - \boldsymbol{x}_0)| &= |\boldsymbol{a} \cdot (\boldsymbol{x} - \boldsymbol{x}_0) + \boldsymbol{a} \cdot (\boldsymbol{y} - \boldsymbol{x})| \\ &\geqslant |\boldsymbol{a} \cdot (\boldsymbol{x} - \boldsymbol{x}_0)| - |\boldsymbol{a} \cdot (\boldsymbol{y} - \boldsymbol{x})| \\ &\geqslant \|\boldsymbol{a}\| \, r - \|\boldsymbol{a}\| \, \|\boldsymbol{y} - \boldsymbol{x}\| \\ &> \|\boldsymbol{a}\| \, r - \|\boldsymbol{a}\| \, r = 0. \end{aligned} \tag{14}$$

Note that we have applied Cauchy-Schwarz inequality. □

Exercise 12. For each of the following properties, find one convex set satisfying it. Justify your answers.

a) Open;

b) Closed;

c) Neither open nor closed;

d) Both open and closed.

Proposition 11. *We have the following:*

a) Union of any number of open sets is open;

b) Intersection of finitely many open sets is open;

c) Union of finitely many closed sets is closed;

d) Intersection of any number of closed sets is closed.

Proof. We prove a) and leave the rest as exercises.

Let W a collection of arbitrary number of open sets. We need to show $E := \cup_{A \in W} A$ is open. Take any $\boldsymbol{x} \in E$. By definition of union, there is $A \in W$ such that $\boldsymbol{x} \in A$. Since A is open, there is $r > 0$ such that $B(\boldsymbol{x}, r) \subseteq A \subseteq E$. Thus ends the proof. □

Exercise 13. Prove b) – d).

Exercise 14. Let A be open and B closed. Prove that $A - B$ is open.

Exercise 15. Construct the following counterexamples. Justify your answers.

a) Find a collection W of open sets such that $\cap_{A \in W} A$ is closed;

b) Find a collection W of open sets such that $\cap_{A \in W} A$ is open;

c) Find a collection W of open sets such that $\cap_{A\in W}A$ is neither open nor closed;

d) Find a collection W of open sets such that $\cap_{A\in W}A$ is both open and closed.

Exercise 16. Often the following definition is given:

> Let $x_0\in\mathbb{R}^n$. $U\subseteq\mathbb{R}^n$ is called a "neighborhood" of $\boldsymbol{x}_0$ if there is an open ball B such that $\boldsymbol{x}_0\in B\subseteq U$.

Prove that a set $E\subseteq\mathbb{R}^n$ is open if and only if E is a neighborhood of each of its points.

Exercise 17. Let $E_1,\ldots,E_N$ be closed (open) sets in $\mathbb{R}$. Prove that $E_1\times\cdots\times E_N$ is closed (open) in $\mathbb{R}^N$.

Problem 4. It is also possible to define open sets through open intervals:

> A set $E\subseteq\mathbb{R}^n$ is *open* if and only if, for any $\boldsymbol{x}\in E$, there an open interval I such that $x\in I\subseteq E$.

Do you think this definition leads to the same topology? Justify your answer.

Problem 5. Let $E\subseteq\mathbb{R}^N$ be open. Prove that

$$E=\cup_{F\subseteq E, F\text{ open}}F. \tag{15}$$

Let $F\subseteq\mathbb{R}^N$ be closed. Prove that

$$F=\cap_{G\supseteq F, G\text{ closed}}G. \tag{16}$$

Limit and continuity in $\mathbb{R}^N$ through open sets

Theorem 12. *Let $\boldsymbol{f}\colon A\subseteq\mathbb{R}^N\mapsto\mathbb{R}^M$. Then $\boldsymbol{f}$ is continuous if and only if*

a) for each open set $U\subseteq\mathbb{R}^M$, $\boldsymbol{f}^{-1}(U)=V\cap A$ for some open set $V\subseteq\mathbb{R}^N$; Or

b) for each closed set $E\subseteq\mathbb{R}^M$, $\boldsymbol{f}^{-1}(E)=F\cap A$ for some closed set $F\subseteq\mathbb{R}^N$.

Proof. We prove a) and leave b) as exercise.

- "Only if". Let $\boldsymbol{f}$ be continuous. Let $U\subseteq\mathbb{R}^M$ be open. If $\boldsymbol{f}(A)\cap U=\varnothing$, then $\boldsymbol{f}^{-1}(U)=\varnothing=\varnothing\cap A$; Otherwise take any $\boldsymbol{y}=\boldsymbol{f}(\boldsymbol{x})\subseteq U$. Since U is open there is $\varepsilon=\varepsilon(\boldsymbol{y})>0$ such that

$$B(\boldsymbol{y},\varepsilon)\subseteq U\Longrightarrow\forall\boldsymbol{z}\in\mathbb{R}^M, \|\boldsymbol{z}-\boldsymbol{y}\|<\varepsilon\text{ then }\boldsymbol{z}\in U. \tag{17}$$

  As $\boldsymbol{f}$ is continuous, there is $\delta=\delta(\boldsymbol{x})>0$ such that for all $\|\boldsymbol{z}-\boldsymbol{x}\|<\delta$, $\|\boldsymbol{f}(\boldsymbol{z})-\boldsymbol{y}\|<\varepsilon$. Clearly we see that this gives $B(\boldsymbol{x},\delta(\boldsymbol{x}))\cap A\subseteq\boldsymbol{f}^{-1}(B(\boldsymbol{y},\varepsilon(\boldsymbol{y})))\subseteq\boldsymbol{f}^{-1}(U)$. Define

$$V:=\cup_{\boldsymbol{x}\in\boldsymbol{f}^{-1}(U)}B(\boldsymbol{x},\delta(\boldsymbol{x})). \tag{18}$$

  Now we have

$$V\cap A=\cup_{\boldsymbol{x}\in\boldsymbol{f}^{-1}(U)}(B(\boldsymbol{x},\delta(\boldsymbol{x}))\cap A)\subseteq\boldsymbol{f}^{-1}(U). \tag{19}$$

  On the other hand cleearly $\boldsymbol{f}^{-1}(U)\subseteq V$. Therefore equality holds.

- "If". Fix any $\boldsymbol{x}_0\in A$, we show that $\boldsymbol{f}$ is continuous at $\boldsymbol{x}_0$. Take any $\varepsilon>0$, set $U=B(\boldsymbol{f}(\boldsymbol{x}_0),\varepsilon)$. Thus we have an open set $V\subseteq\mathbb{R}^N$ such that $\boldsymbol{f}^{-1}(U)=V\cap A$ or equivalently $\boldsymbol{f}(V\cap A)\subseteq U$. Since V is open, there is $\delta>0$ such that $B(\boldsymbol{x}_0,\delta)\subseteq V$.

Continuity follows as $\boldsymbol{f}(B(\boldsymbol{x}_0, \delta) \cap A) \subseteq B(\boldsymbol{f}(\boldsymbol{x}_0, \varepsilon))$ is equivalent to

$$\forall \boldsymbol{x} \text{ with } \|\boldsymbol{x} - \boldsymbol{x}_0\| < \delta, \qquad \|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x}_0)\| < \varepsilon. \tag{20}$$

Thus the proof ends. □

Exercise 18. From $\boldsymbol{f}^{-1}(U) = V \cap A$, can we conclude $\boldsymbol{f}(V \cap A) = U$? Justify.

Exercise 19. Prove b).