# Definitions and Properties of $\mathbb{R}^N$

## $\mathbb{R}^N$ as a set

As a set $\mathbb{R}^n$ is simply the set of all ordered $n$-tuples $(x_1, ..., x_N)$, called "vectors". We usually denote the vector $(x_1, ..., x_N), (y_1, ..., y_N), ...$ by $\boldsymbol{x}, \boldsymbol{y}, ...$ or $\vec{x}, \vec{y}, ...$. For a vector $\boldsymbol{x} \in \mathbb{R}^N$, the numbers $x_1, ..., x_N$ are called its 1st, 2nd, ..., $N$-th coordinates.

**Remark 1.** There are two ways to write a vector: as a row $(x_1, ..., x_N)$ or as a column $\begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}$. In the following we will see that it is often more convenient to write vectors as rows when discussing $\mathbb{R}^N$ itself, but as columns when discussing functions on $\mathbb{R}^N$.

**Remark 2. (Cartesian product)** Let $A, B$ be two sets. Their Cartesian product is defined as the new set consisting of of ordered pairs: $A \times B := \{(x, y) | x \in A, y \in B\}$. Similarly, for finitely many sets $A_1, ..., A_m$, their Cartesian product is defined as

$$A_1 \times \cdots \times A_m := \{(x_1, ..., x_m) | x_i \in A_i, \ i = 1, 2, ..., m\}. \tag{1}$$

Thus we see that $\mathbb{R}^N = \mathbb{R} \times \cdots \times \mathbb{R}$ ($N$ times).

**Remark 3.** Note that Cartesian product is not commutative. That is $A \times B \neq B \times A$.

**Exercise 1.** Give a sufficient and necessary condition on $A, B$ for $A \times B = B \times A$.

## $\mathbb{R}^N$ as a linear vector space

On $\mathbb{R}^N$ one can define the operations of addition and scalar multiplication:

- Addition:

$$\boldsymbol{x} + \boldsymbol{y} = (x_1, ..., x_N) + (y_1, ..., y_N) := (x_1 + y_1, ..., x_N + y_N); \tag{2}$$

- Scalar multiplication: Let $a \in \mathbb{R}$, then

$$a\,\boldsymbol{x} = a \cdot (x_1, ..., x_N) := (a\,x_1, ..., a\,x_N); \tag{3}$$

The set $\mathbb{R}^N$ equipped with these two operations becomes a "real linear vector space".

**Definition 4. (Real linear vector space)** *A set $X$ is called a (real) linear vector space if two operations $\oplus$ and $\odot$ and defined satisfying the following properties: Let $x, y, z, ... \in X$ and $a, b, c, ... \in \mathbb{R}$.*

*i.* $x \oplus y = y \oplus x$;

*ii.* $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;

*iii. There is an element $x_0 \in X$ such that $x_0 \oplus x = x$ for every $x \in X$. We denote $x_0$ by $0$;*

*iv. For every $x \in X$ there is $y \in X$ such that $x \oplus y = 0$. We denote $y$ by $(-x)$;*

*v.* $a \odot (x \oplus y) = (a \odot x) \oplus (a \odot y)$;

*vi.* $(a+b)\odot x=(a\odot x)+(b\odot x)$*;*

*vii.* $(a\,b)\odot x=a\odot(b\odot x)$*;*

*viii.* $1\odot x=x$*.*

**Example 5.** Prove that $\mathbb{R}^N$ is a linear vector space.

**Proof.** The proofs are straightforward. We only show the details of part iv here and leave the rest as exercises.

- Proof of part iv. For any $\boldsymbol{x}\in\mathbb{R}^N$, let its coordinates be $x_1,\ldots,x_N$. Now define the vector $\boldsymbol{y}:=\begin{pmatrix}-x_1\\ \vdots\\ -x_N\end{pmatrix}$. We easily check that $\boldsymbol{x}+\boldsymbol{y}=\boldsymbol{0}$. □

**Exercise 2.** Consider the set $X$ of infinite real sequences. Define

$$\{x_n\}\oplus\{y_n\}:=\{x_n+y_n\};\qquad a\odot\{x_n\}=\{a\,x_n\}. \tag{4}$$

Prove that $X$ is now a real linear vector space.

**Exercise 3.** Consider the set of all functions $f:[0,1]\mapsto\mathbb{R}$. Define $\oplus,\odot$ appropriately to make it a real linear vector space.

We notice that many natural properties are not listed in Definition 4. They can all be derived from i) – viii). In the following we illustrate how this could be done through proving $0\odot x=0$ for all $x\in X$ a real linear vector space.

**Exercise 4.** In $0\odot x=0$, do the two 0's mean the same thing?

**Lemma 6.** *Let $X$ be a real linear vector space, then the element $0$ is unique.*

**Proof.** We prove by contradiction. Let $0_1\neq 0_2$ be two zero elements. Then we have

$$0_1=0_1\oplus 0_2=0_2. \tag{5}$$

Contradiction. □

**Lemma 7.** *Let $X$ be a real linear vector space and $x\in X$. Then $-x$ is unique.*

**Proof.** Prove by contradiction. Let $y,z$ be such that $x\oplus y=x\oplus z=0$. Then we have

$$z=(x\oplus y)\oplus z=y\oplus(x\oplus z)=y\oplus 0=y. \tag{6}$$

Thus ends the proof. □

**Lemma 8. (Cancellation)** *Let $X$ be a real linear vector space and $x,y,z\in X$. If $x\oplus y=x\oplus z$, then $y=z$.*

**Proof.** We have

$$y=(-x)\oplus(x\oplus y)=(-x)\oplus(x\oplus z)=[(-x)\oplus x]\oplus z=0\oplus z=z. \tag{7}$$

Thus ends the proof. □

**Theorem 9.** *Let $X$ be a real linear vector space, then $0 \odot x = 0$.*

**Proof.** We have

$$0 \odot x + 0 \odot x = (0 + 0) \odot x = 0 \odot x \tag{8}$$

Cancelling one $0 \odot x$ (thanks to the previous lemma) we have $0 \odot x = 0$. □

**Exercise 5.** Let $X$ be a real linear vector space. Prove that $(-1) \odot x = -x$. Note that here the $-$ and $x$ in $-x$ shouldn't be understood separately. $-x$ is just an intuitive notation for the vector $y$ satisfying $y \oplus x = 0$.

### $\mathbb{R}^n$ as an inner product space

For the development of linear algebra theory, linear vector space is enough. However if we would like to do geometry, then we need notions such as angle and length. This can be done through the definition of one more operation: inner product.

**Definition 10.** *The* inner product *on $\mathbb{R}^n$ is defined through*

$$\boldsymbol{x} \cdot \boldsymbol{y} = (x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) := (x_1\, y_1, \ldots, x_n\, y_n). \tag{9}$$

**Lemma 11.** *Let $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^n$ and let $a, b \in \mathbb{R}$.*

*a)* $\boldsymbol{x} \cdot \boldsymbol{x} \geqslant 0$. $\boldsymbol{x} \cdot \boldsymbol{x} = 0 \Longleftrightarrow \boldsymbol{x} = \mathbf{0}$.

*b)* $\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{y} \cdot \boldsymbol{x}$;

*c)* $(a\, \boldsymbol{x} + b\, \boldsymbol{y}) \cdot \boldsymbol{z} = a\, (\boldsymbol{x} \cdot \boldsymbol{z}) + b\, (\boldsymbol{y} \cdot \boldsymbol{z})$.

**Proof.** Left as exercise. □

**Remark 12. (Abstract inner product)** An inner product on an abstract real linear vector space $X$ is a function mapping any $x, y, z \in X$ to a number in $\mathbb{R}$, denoted $(x, y)$, satisfying the following.

a) (Positive definiteness) $(x, x) \geqslant 0$; $(x, x) = 0 \Longleftrightarrow x = 0$;

b) (Conjugate symmetry) $(x, y) = (y, x)$;

c) (Linearity) $(a\, x, y) = a\, (x, y)$; $(x + y, z) = (x, z) + (y, z)$.

**Exercise 6.** Consider the space of infinite sequences $\{x_n\}$. How would you define its inner product? Is it possible to define inner product for all sequences?

**Exercise 7.** Consider the space of functions $f: [0, 1] \mapsto \mathbb{R}$. How would you define its inner product? Is it possible to define inner product for all such functions?

### Norm and distance

**Definition 13. (Euclidean norm)** *Let $\boldsymbol{x} = (x_1, \ldots, x_N)$ be a vector in $\mathbb{R}^N$. Then we define its Euclidean norm by*

$$\|\boldsymbol{x}\| := (x_1^2 + \cdots + x_N^2)^{1/2}. \tag{10}$$

**Exercise 8.** Let $\boldsymbol{x} \in \mathbb{R}^N$. Show that $\|\boldsymbol{x}\| = (\boldsymbol{x} \cdot \boldsymbol{x})^{1/2}$.

**Lemma 14.** *Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$. Then*

$$\boldsymbol{x} \cdot \boldsymbol{y} = \frac{1}{4} (\|\boldsymbol{x} + \boldsymbol{y}\|^2 - \|\boldsymbol{x} - \boldsymbol{y}\|^2). \tag{11}$$

**Proof.** We have by definition

$$\|\boldsymbol{x} + \boldsymbol{y}\|^2 = (\boldsymbol{x} + \boldsymbol{y}) \cdot (\boldsymbol{x} + \boldsymbol{y}), \qquad \|\boldsymbol{x} - \boldsymbol{y}\|^2 = (\boldsymbol{x} - \boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y}). \tag{12}$$

Expanding the right hand sides and the conclusion follows. □

**Lemma 15.** *We have the following. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$ and $a \in \mathbb{R}$.*

*a)* $\|\boldsymbol{x}\| \geqslant 0$ *for all* $\boldsymbol{x} \in \mathbb{R}^N$, *and* $\|\boldsymbol{x}\| = 0 \Longleftrightarrow \boldsymbol{x} = 0$;

*b)* $\|a\, \boldsymbol{x}\| = |a|\, \|\boldsymbol{x}\|$;

*c)* **(Triangle inequality)** $\|\boldsymbol{x} + \boldsymbol{y}\| \leqslant \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$.

**Proof.** The only non-trivial claim is c). To prove we square both sides:

$$\|\boldsymbol{x} + \boldsymbol{y}\|^2 = (x_1 + y_1)^2 + \cdots + (x_N + y_N)^2; \tag{13}$$

$$(\|\boldsymbol{x}\| + \|\boldsymbol{y}\|)^2 = (x_1^2 + \cdots + x_N^2) + (y_1^2 + \cdots + y_N^2) + 2\, (x_1^2 + \cdots + x_N^2)^{1/2}\, (y_1^2 + \cdots + y_N^2)^{1/2}. \tag{14}$$

Therefore all we need to do is to prove

$$x_1\, y_1 + \cdots + x_N\, y_N \leqslant (x_1^2 + \cdots + x_N^2)^{1/2}\, (y_1^2 + \cdots + y_N^2)^{1/2} \tag{15}$$

Taking square of both sides and we see that this is equivalent to

$$\sum_{i,j=1}^{N} (x_i\, y_i - x_j\, y_j)^2 \geqslant 0. \tag{16}$$

Thus ends the proof. □

**Exercise 9. (Abstract norm)** Let $X$ be a linear vector space. A norm on $X$ is a function $X \mapsto \mathbb{R}$ satisfying a) – c) from Lemma 15. Prove that the following are both norms on $\mathbb{R}^N$:

$$\|\boldsymbol{x}\|_\infty := \max_{i=1,\ldots,N} \{|x_i|\}; \qquad \|\boldsymbol{x}\|_1 := |x_1| + |x_2| + \cdots + |x_N|; \tag{17}$$

**Exercise 10.** Let $X$ be a linear vector space with norm $\|\cdot\|$. Prove the following: If one can define an inner product $(\cdot, \cdot)$ such that $\|x\| = (x, x)^{1/2}$, then for any $x, y \in X$,

$$\|x + y\|^2 + \|x - y\|^2 = 2\, (\|x\|^2 + \|y\|^2). \tag{18}$$

Explain this result using a parallelogram. Then find a norm on $\mathbb{R}^n$ that cannot be defined through an inner product.

**Theorem 16. (Cauchy-Schwarz Inequality)** *Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$. Then*

$$|\boldsymbol{x} \cdot \boldsymbol{y}| \leqslant \|\boldsymbol{x}\|\, \|\boldsymbol{y}\|. \tag{19}$$

**Proof.** This follows immediately from (15). ☐

**Exercise 11.** Prove Cauchy-Schwarz using the following idea: For any $t \in \mathbb{R}$,

$$\|\boldsymbol{x} - t\,\boldsymbol{y}\|^2 \geqslant 0. \tag{20}$$

**Exercise 12.** Let $X$ be an abstract linear vector space with inner product $(\cdot, \cdot)$. Define "norm"

$$\|x\| := (x, x)^{1/2}. \tag{21}$$

a) Prove that the conclusions of Lemma 15 still hold.

b) Prove that Cauchy-Schwarz still hold.

**Definition 17.** *Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$. Their distance is defined as $d(\boldsymbol{x}, \boldsymbol{y}) := \|\boldsymbol{x} - \boldsymbol{y}\|$.*

**Exercise 13.** Explain Triangle Inequality using a triangle with vertex $\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3} \in \mathbb{R}^N$;

**Exercise 14.** Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{\boldsymbol{N}}$. Prove

$$d(\boldsymbol{x}, \boldsymbol{y}) := \|\boldsymbol{x} - \boldsymbol{y}\| \leqslant |\|\boldsymbol{x}\| - \|\boldsymbol{y}\||. \tag{22}$$

Explain what this means geometrically.

**Definition 18. (Distance between sets)** *Let $E, F \subseteq \mathbb{R}^N$. Their distance* $\operatorname{dist}(E, F)$ *is defined as*

$$\operatorname{dist}(E, F) := \inf_{\boldsymbol{x} \in E, \boldsymbol{y} \in F} \operatorname{dist}(\boldsymbol{x}, \boldsymbol{y}) = \inf_{\boldsymbol{x} \in E, \boldsymbol{y} \in F} \|\boldsymbol{x} - \boldsymbol{y}\|, \tag{23}$$

**Exercise 15.** Find two sets $E, F \subseteq \mathbb{R}^N$ such that $\operatorname{dist}(E, F) = 0$ but $E \cap F = \varnothing$.

**Exercise 16.** Let $\boldsymbol{x} \in \mathbb{R}^2$, $E \subset \mathbb{R}^2$. Find $\operatorname{dist}(\boldsymbol{x}, E)$.

a) $\boldsymbol{x} = (2, 2)$; $E = \{(x, y) |\, 0 < x < 1, 0 < y < 1\}$;

b) $\boldsymbol{x} = (2, 1)$; $E = \{(x, y) |\, x^2 + y^2 \leqslant 2\}$;

c) $\boldsymbol{x} = (0, 0)$; $E = \{(x, y) |\, 1 \leqslant x^2 + y^2 \leqslant 2\}$.

Justify your answers.

**Exercise 17.** Let $\boldsymbol{x} \in \mathbb{R}^N$ and $E, F \subseteq \mathbb{R}^N$ nonempty. Prove

a) $\boldsymbol{x} \in E \Longrightarrow \operatorname{dist}(\boldsymbol{x}, E) = 0$;

b) $E \subseteq F \Longrightarrow \operatorname{dist}(\boldsymbol{x}, E) \geqslant \operatorname{dist}(\boldsymbol{x}, F)$;

c) $\operatorname{dist}(\boldsymbol{x}, E \cup F) = \min\{\operatorname{dist}(\boldsymbol{x}, E), \operatorname{dist}(\boldsymbol{x}, F)\}$.

Can we say anything about $\operatorname{dist}(\boldsymbol{x}, E \cap F)$?

**Exercise 18.** Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$ and $E \subseteq \mathbb{R}^N$ nonempty. Prove

$$\operatorname{dist}(\boldsymbol{x}, E) \leqslant \operatorname{dist}(\boldsymbol{x}, \boldsymbol{y}) + \operatorname{dist}(\boldsymbol{y}, E). \tag{24}$$

**Exercise 19.** Let $E, F, G \subseteq \mathbb{R}^N$ be nonempty. Do we have

$$\operatorname{dist}(E, F) \leqslant \operatorname{dist}(E, G) + \operatorname{dist}(F, G)? \tag{25}$$

Justify your answer.

## Angle

**Lemma 19.** *Let* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$. *Then* $\boldsymbol{x} \perp \boldsymbol{y} \Longleftrightarrow \boldsymbol{x} \cdot \boldsymbol{y} = 0$.

**Proof.** $\Longrightarrow$. If $\boldsymbol{x} \perp \boldsymbol{y}$, then by Pythagorean Theorem we have $\|\boldsymbol{x} - \boldsymbol{y}\|^2 = \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2$. This gives

$$(\boldsymbol{x} - \boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y}) = \boldsymbol{x} \cdot \boldsymbol{x} + \boldsymbol{y} \cdot \boldsymbol{y} \Longrightarrow \boldsymbol{x} \cdot \boldsymbol{y} = 0. \tag{26}$$

The other direction is left as exercise. □

**Remark 20.** Note that here we choose to define "perpendicular" through Pythagorean Theorem.

**Exercise 20.** Prove that $\boldsymbol{x} \cdot \boldsymbol{y} = 0 \Longrightarrow \boldsymbol{x} \perp \boldsymbol{y}$.

**Exercise 21.** Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$. Prove that

$$\|\boldsymbol{x}\| = \|\boldsymbol{y}\| \Longleftrightarrow (\boldsymbol{x} - \boldsymbol{y}) \cdot (\boldsymbol{x} + \boldsymbol{y}) = 0. \tag{27}$$

Explain the geometrical meaning of this result.

**Definition 21.** *Let* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$, *define the angle* $\theta$ *by*

$$\theta = \arccos\left(\frac{\boldsymbol{x} \cdot \boldsymbol{y}}{\|\boldsymbol{x}\| \, \|\boldsymbol{y}\|}\right). \tag{28}$$

**Exercise 22.** Explain why $\theta$ is defined for all pairs of vectors.