Applications of single variable calculus: Additive functions

Positive results

Definition 1. *Let $f: \mathbb{R} \mapsto \mathbb{R}$. f is said to be "additive" if the following holds:*

$$\forall x, y \in \mathbb{R} \qquad f(x+y) = f(x) + f(y). \tag{1}$$

Remark 2. The above functional equations is called the "Cauchy equation".

Exercise 1. Assume f to be continuous. We try to prove that $f(x) = a\,x$ for some $a \in \mathbb{R}$ through the following steps.

a) Prove that $f(0) = 0$;

b) Give a reasonable guess of a;

c) Prove that there is $a \in \mathbb{R}$ such that $f(m) = a\,m$ for all $m \in \mathbb{Z}$ (that is m is an integer);

d) Prove that for the same a, $f(x) = a\,x$ for all $x \in \mathbb{Q}$ (rational);

e) Prove $f(x) = a\,x$ through continuity.

Exercise 2. A more clever proof using fundamental theorem of calculus can be carried out as follows.

a) Prove through integrating (1) to obtain

$$f(x) = \int_x^{1+x} f(x)\,\mathrm{d}x - \int_0^1 f(y)\,\mathrm{d}y. \tag{2}$$

b) Differentiate with respect to x to reach the conclusion.

c) Explain how continuity is used in this proof.

Exercise 3. Assume f to be merely locally integrable (that is for any finite interval $[a, b]$, $f(x)$ is integrable on it). Prove that if f is additive then it must be of the form $f(x) = a\,x$. (Hint: Integrate

$$x'\,f(x) = \int_0^{x'} f(x)\,\mathrm{d}y \tag{3}$$

and observe symmetry.[1])

Remark 3. The assumption on f can be further relaxed to only Lebesgue measurable.[2] [3] This is beyond our course and we will not discuss the proof. However, an important observation in Banach's proof is the following.

Exercise 4. Let $f(x)$ be additive and $x_0 \in \mathbb{R}$. Assume $f(x)$ is continuous at x_0. Prove that $f(x)$ is continuous everywhere.

Nonlinear additive functions

Are there nonlinear additive functions? If there are, then they are very weird looking beings.

1. This proof is due to H. N. Shapiro: *A Micronote on a functional equation*, The American Mathematical Monthly, Vol. 80, No. 9, p.1041, 1973.

2. S. Banach, *Sur l'équation fonctionnelle* $f(x + y) = f(x) + f(y)$, Fundamenta Mathematicae. 1 (1920), available at http://matwbn.icm.edu.pl/ksiazki/fm/fm1/fm1115.pdf.

3. W. Siepinski, *Sur l'équation fonctionnelle* $f(x + y) = f(x) + f(y)$, Fundamenta Mathematicae. 1 (1920), available at http://matwbn.icm.edu.pl/ksiazki/fm/fm1/fm1114.pdf.

Theorem 4. *If f is additive but not linear, then the graph of f is dense in $\mathbb{R}^2$.*

Remark 5. Here the graph of a function is the set of points $\{(x, f(x))|\, x \in \mathbb{R}\}$ in the plain $\mathbb{R}^2$. And "dense" means for any point $(u, v) \in \mathbb{R}^2$ and any $\varepsilon > 0$, there is $x \in \mathbb{R}$ such that $(x, f(x))$ lies inside the circle with center (u, v) and radius ε.

Proof. First observe that f is linear $\Longleftrightarrow$ $f(0) = 0$ and $\forall x, y \in \mathbb{R}, x, y \neq 0$, $\frac{f(x)}{x} = \frac{f(y)}{y}$. From a previous exercise we know that f is additive $\Longrightarrow$ $f(0) = 0$. Therefore if f is a nonlinear additive function, then there are $x_1, x_2 \in \mathbb{R}$ such that

$$\frac{f(x_1)}{x_1} \neq \frac{f(x_2)}{x_2}. \tag{4}$$

This is equivalent to the linear independence of the two vectors $\begin{pmatrix} x_1 \\ f(x_1) \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ f(x_2) \end{pmatrix}$. The independence implies that for any vector $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2$, there are $a, b \in \mathbb{R}$ such that

$$\begin{pmatrix} u \\ v \end{pmatrix} = a \begin{pmatrix} x_1 \\ f(x_1) \end{pmatrix} + b \begin{pmatrix} x_2 \\ f(x_2) \end{pmatrix}. \tag{5}$$

The proof ends once we prove the following claim: For any $a, b \in \mathbb{Q}$ (rational numbers), there are $x \in \mathbb{R}$ such that

$$\begin{pmatrix} x \\ f(x) \end{pmatrix} = a \begin{pmatrix} x_1 \\ f(x_1) \end{pmatrix} + b \begin{pmatrix} x_2 \\ f(x_2) \end{pmatrix}. \tag{6}$$

The proof of this claim is left as exercise. □

Exercise 5. Prove the claim at the end of the above proof.

Exercise 6. Prove that if f is additive and satisfies any of the following, then it is linear.

a) f is bounded from above.

b) f is monoton.

c) f is convex.

Remark 6. This is kind of a universal phenomenon in real analysis: A function is either super nice or horrible. The deep reason for this is the existence depends on the so-called Axiom of Choice: Let X be a collection of sets, then there is a function $f: X \mapsto \cup_{A \in X} A$ such that $f(A) \in A$. In every day language, given any collection of sets, there is always a rule to pick one element from each set. Accepting this axiom or not has profound implications. Two examples: Banach-Tarski paradox[4] and existence of Lebesgue unmeasurable sets.[5]

The construction of such functions was done by Georg Hamel (1877-1954)[6] using the existence of so-called Hamel basis (which of course depends on Axiom of Choice)

4. Given a ball in the space $\mathbb{R}^3$, you can break it into finitely many pieces and re-assemble these pieces to obtain two balls, each identical to the original ball.

5. It's been proved by Robert M. Solovay (1938 –) (*A model of set-theory in which every set of reals is Lebesgue measurable*, Ann. Math. 92 1–56 1970) that, every set is Lebesgue measurable if we decide to not accept Axiom of Choice.

6. *Eine Basis aller Zahlen und die unstetigen Losungen der Functionalgleichung:* $f(x+y) = f(x) + f(y)$, Math. Ann. 60 459-462 1905.

Definition 7. (Hamel basis on $\mathbb{R}$) *A Hamel basis of $\mathbb{R}$ is a subset B of $\mathbb{R}$ such that every $x \in \mathbb{R}$ has a unique representation as a finite linear combination of numbers in B with rational coefficients.*

Exercise 7. Prove that if $x \in \mathbb{R}$ can be written as $s = a \cdot 1 + b\sqrt{2} + c\sqrt{3}$ with $a, b, c \in \mathbb{Q}$, then such representation is unique.

Exercise 8. Prove that $\{1, \sqrt{2}, \sqrt{3}\}$ is not a Hamel basis of $\mathbb{R}$.

Exercise 9. Let f_1, f_2 be additive and let B be a Hamel basis of $\mathbb{R}$. Then $f_1 = f_2$ on $B \Longrightarrow f_1 = f_2$ on $\mathbb{R}$.

Exercise 10. Given the existence of a Hamel basis B for $\mathbb{R}$, Construct a nonlinear additive function.