Applications of single variable calculus: Additive functions

Positive results

Definition 1. Let $f: \mathbb{R} \to \mathbb{R}$. $f$ is said to be “additive” if the following holds:

$$\forall x, y \in \mathbb{R} \quad f(x + y) = f(x) + f(y). \quad (1)$$

Remark 2. The above functional equations is called the “Cauchy equation”.

Exercise 1. Assume $f$ to be continuous. We try to prove that $f(x) = a \cdot x$ for some $a \in \mathbb{R}$ through the following steps.

a) Prove that $f(0) = 0$;

b) Give a reasonable guess of $a$;

c) Prove that there is $a \in \mathbb{R}$ such that $f(m) = a \cdot m$ for all $m \in \mathbb{Z}$ (that is $m$ is an integer);

d) Prove that for the same $a$, $f(x) = a \cdot x$ for all $x \in \mathbb{Q}$ (rational);

e) Prove $f(x) = a \cdot x$ through continuity.

Exercise 2. A more clever proof using fundamental theorem of calculus can be carried out as follows.

a) Prove through integrating (1) to obtain

$$f(x) = \int_0^{1+x} f(x) \, dx - \int_0^1 f(y) \, dy. \quad (2)$$

b) Differentiate with respect to $x$ to reach the conclusion.

c) Explain how continuity is used in this proof.

Exercise 3. Assume $f$ to be merely locally integrable (that is for any finite interval $[a, b]$, $f(x)$ is integrable on it). Prove that if $f$ is additive then it must be of the form $f(x) = a \cdot x$. (Hint: Integrate

$$x' \cdot f(x) = \int_0^{x'} f(x) \, dy \quad (3)$$
and observe symmetry.)

Remark 3. The assumption on $f$ can be further relaxed to only Lebesgue measurable. This is beyond our course and we will not discuss the proof. However, an important observation in Banach’s proof is the following.

Exercise 4. Let $f(x)$ be additive and $x_0 \in \mathbb{R}$. Assume $f(x)$ is continuous at $x_0$. Prove that $f(x)$ is continuous everywhere.

Nonlinear additive functions

Are there nonlinear additive functions? If there are, then they are very weird looking beings.

Theorem 4. If \( f \) is additive but not linear, then the graph of \( f \) is dense in \( \mathbb{R}^2 \).

Remark 5. Here the graph of a function is the set of points \( \{(x, f(x)) \mid x \in \mathbb{R}\} \) in the plain \( \mathbb{R}^2 \). And “dense” means for any point \((u, v) \in \mathbb{R}^2\) and any \( \varepsilon > 0 \), there is \( x \in \mathbb{R} \) such that \( (x, f(x)) \) lies inside the circle with center \((u, v)\) and radius \( \varepsilon \).

Proof. First observe that \( f \) is linear \( \iff f(0) = 0 \) and \( \forall x, y \in \mathbb{R}, x, y \neq 0, \frac{f(x)}{x} = \frac{f(y)}{y} \). From a previous exercise we know that \( f \) is additive \( \implies f(0) = 0 \). Therefore if \( f \) is a nonlinear additive function, then there are \( x_1, x_2 \in \mathbb{R} \) such that

\[
\frac{f(x_1)}{x_1} \neq \frac{f(x_2)}{x_2}. \tag{4}
\]

This is equivalent to the linear independence of the two vectors \( \left( \frac{x_1}{f(x_1)} \right) \) and \( \left( \frac{x_2}{f(x_2)} \right) \). The independence implies that for any vector \( \left( \frac{u}{v} \right) \in \mathbb{R}^2 \), there are \( a, b \in \mathbb{R} \) such that

\[
\left( \frac{u}{v} \right) = a \left( \frac{x_1}{f(x_1)} \right) + b \left( \frac{x_2}{f(x_2)} \right). \tag{5}
\]

The proof ends once we prove the following claim: For any \( a, b \in \mathbb{Q} \) (rational numbers), there are \( x \in \mathbb{R} \) such that

\[
\left( \frac{x}{f(x)} \right) = a \left( \frac{x_1}{f(x_1)} \right) + b \left( \frac{x_2}{f(x_2)} \right). \tag{6}
\]

The proof of this claim is left as exercise. \( \square \)

Exercise 5. Prove the claim at the end of the above proof.

Exercise 6. Prove that if \( f \) is additive and satisfies any of the following, then it is linear.

a) \( f \) is bounded from above.

b) \( f \) is monoton.

c) \( f \) is convex.

Remark 6. This is kind of a universal phenomenon in real analysis: A function is either super nice or horrible. The deep reason for this is the existence depends on the so-called Axiom of Choice: Let \( X \) be a collection of sets, then there is a function \( f: X \mapsto \bigcup_{A \in X} A \) such that \( f(A) \in A \). In every day language, given any collection of sets, there is always a rule to pick one element from each set. Accepting this axiom or not has profound implications. Two examples: Banach-Tarski paradox\textsuperscript{4} and existence of Lebesgue unmeasurable sets\textsuperscript{5}.

The construction of such functions was done by Georg Hamel (1877-1954)\textsuperscript{6} using the existence of so-called Hamel basis (which of course depends on Axiom of Choice).

\textsuperscript{4} Given a ball in the space \( \mathbb{R}^3 \), you can break it into finitely many pieces and re-assemble these pieces to obtain two balls, each identical to the original ball.

\textsuperscript{5} It’s been proved by Robert M. Solovay (1938- ) (\textit{A model of set-theory in which every set of reals is Lebesgue measurable}, Ann. Math. 92 1–56 1970) that, every set is Lebesgue measurable if we decide to not accept Axiom of Choice.

\textsuperscript{6} \textit{Eine Basis aller Zahlen und die unstetigen Losungen der Functionalgleichung: }\( f(x + y) = f(x) + f(y) \), Math. Ann. 60 459-462 1905.
Definition 7. (Hamel basis on \( \mathbb{R} \)) A Hamel basis of \( \mathbb{R} \) is a subset \( B \) of \( \mathbb{R} \) such that every \( x \in \mathbb{R} \) has a unique representation as a finite linear combination of numbers in \( B \) with rational coefficients.

Exercise 7. Prove that if \( x \in \mathbb{R} \) can be written as \( s = a \cdot 1 + b \sqrt{2} + c \sqrt{3} \) with \( a, b, c \in \mathbb{Q} \), then such representation is unique.

Exercise 8. Prove that \( \{1, \sqrt{2}, \sqrt{3}\} \) is not a Hamel basis of \( \mathbb{R} \).

Exercise 9. Let \( f_1, f_2 \) be additive and let \( B \) be a Hamel basis of \( \mathbb{R} \). Then \( f_1 = f_2 \) on \( B \Rightarrow f_1 = f_2 \) on \( \mathbb{R} \).

Exercise 10. Given the existence of a Hamel basis \( B \) for \( \mathbb{R} \), Construct a nonlinear additive function.