# Properties of Riemann integrals

## Fundamental Theorem of Calculus

**Theorem 1. (FTC 1st Version)** *Let $f$ be integrable on $[a, b]$. If $F$ is continuous on $[a, b]$ and is an antiderivative of $f$, that is $F' = f$, on $(a, b)$, then*

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a). \tag{1}$$

**Exercise 1.** What is the significance of this theorem? How did it help you calculating integrals?

**Exercise 2.** Explain why we assume "$F$ is continuous on $[a, b]$". Isn't it already a consequence of the differentiability of $F$?

**Exercise 3.** Find a function $f: [0, 1] \mapsto \mathbb{R}$ that is integrable but has no antiderivative.

**Theorem 2. (FTC 2nd Version)** *Let $f$ be integrable on $[a, b]$. Then $G(x) := \int_a^x f(t)\,\mathrm{d}t$ is continuous on $[a, b]$. Furthermore if $f$ is continuous at a point $x_0 \in (a, b)$, then $G$ is differentiable at $x_0$ and $G'(x_0) = f(x_0)$.*

**Exercise 4.** Let $f$ be continuous on $[a, b]$. Calculate

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{-\cos(x)}^{\exp(x)} f(t^2)\,\mathrm{d}t\right). \tag{2}$$

**Exercise 5.** Find a function $f$ that is integrable on $[a, b]$ with $G(x) := \int_a^x f(t)\,\mathrm{d}t$ differentiable but $G' \neq f$.

## Change of variables

**Theorem 3.** *Let $f(x)$ be continuous on $[a, b]$, $\varphi(t): [\alpha, \beta] \mapsto \mathbb{R}$ is continuous on $[\alpha, \beta]$ and differentiable with $\varphi'(t)$ is integrable. Further assume $\varphi(\alpha) = a, \varphi(\beta) = b, \varphi([\alpha, \beta]) \subseteq [a, b]$. Then*

$$\int_a^b f(x)\,\mathrm{d}x = \int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,\mathrm{d}t. \tag{3}$$

**Remark 4.** Note that we don't need $\varphi$ to be one-to-one! However see the following exercise.

**Exercise 6.** Let $f$ be integrable on $[a, b]$, $\varphi$ satisfies:

i. $\varphi$ is continuous and differentiable on $[a, b]$;

ii. $\varphi'$ is continuous on $[a, b]$;

iii. $\varphi(\alpha) = a, \varphi(\beta) = b$;

iv. $\varphi$ is strictly monotone.

Then

$$\int_a^b f(x)\,\mathrm{d}x = \int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,\mathrm{d}t. \tag{4}$$

(Hint: Use Riemann sum.)

**Exercise 7.** Explore whether monotonicity is necessary in the above exercise.

**Intermediate value theorems**

**Theorem 5. (First intermediate value theorem)** *Let $f, g$ be integrable on $[a,b]$ and $g$ does not change sign on $[a,b]$. Then there is $s \in [\min_{[a,b]} f, \max_{[a,b]} f]$ such that*

$$\int_a^b f(x)\, g(x)\, \mathrm{d}x = s \int_a^b g(x)\, \mathrm{d}x. \tag{5}$$

*If $f$ is continuous on $[a,b]$, then there is $\xi \in [a,b]$ such that $f(\xi) = s$.*

**Exercise 8.** Prove the theorem.

**Exercise 9.** Does the theorem still hold if we remove the sign condition on $g$? Justify your answer.

**Exercise 10.** Let $f, g$ be integrable on $[a,b]$ and $g$ does not change sign on $[a,b]$. Assume there is $F(x)$ such that $F'(x) = f(x)$ on $(a,b)$. Can we conclude the existence of $\xi \in (a,b)$ such that

$$\int_a^b f(x)\, g(x)\, \mathrm{d}x = f(\xi) \int_a^b g(x)\, \mathrm{d}x? \tag{6}$$

Justify your answer.

**Theorem 6. (Second intermediate value theorem)** *Let $f: [a, b] \mapsto \mathbb{R}$ be integrable, and $g: [a, b] \mapsto \mathbb{R}$ satisfying $g \geqslant 0$. Then*

*a) If furthermore $g$ is decreasing, then there is $\xi \in [a,b]$ such htat*

$$\int_a^b f(x)\, g(x)\, \mathrm{d}x = g(a) \int_a^\xi f(x)\, \mathrm{d}x; \tag{7}$$

*b) If furthermore $g$ is increasing, then there is $\xi \in [a,b]$ such that*

$$\int_a^b f(x)\, g(x)\, \mathrm{d}x = g(b) \int_\xi^b f(x)\, \mathrm{d}x; \tag{8}$$

*c) If only assume furthermore that $g$ is monotone, then there is $\xi \in [a,b]$ such that*

$$\int_a^b f(x)\, g(x)\, \mathrm{d}x = g(a) \int_a^\xi f(x)\, \mathrm{d}x + g(b) \int_\xi^b f(x)\, \mathrm{d}x. \tag{9}$$

**Remark 7.** Note that the integrability of $f g$ is not assumed!

**Exercise 11.** Prove the theorem in the case $f(x)$ doesn't change sign either. (Hint: Define $H(y) := g(a) \int_a^y f(x)\, \mathrm{d}x + g(b) \int_y^b f(x)\, \mathrm{d}x$ and try to use intermediate value theorem for continuous functions)