Riemann integration

Partition and integration of step functions

- Integration theory is developed through the effort of making the ideas of "area" "volume" exact.

Definition 1. (Partition) *Let $a,\ b \in \mathbb{R},\ a < b$. A partition of the interval $[a,\ b]$ is a set of points $P = \{x_0, \ldots, x_n\}$ such that $a = x_0 < x_1 < \cdots < x_n = b$.*

Definition 2. (Refinement) *A refinement of a partition P is a partition $Q \supseteq P$. In this case we also say Q is finer than P.*

Definition 3. (Step function) *A function $f\colon [a,\ b] \mapsto \mathbb{R}$ is called a step function if there is a partition $P = \{x_0, \ldots, x_n\}$ such that $f(x)$ is constant on every (x_{j-1}, x_j).*

Remark 4. Note that there is some ambiguity in the definition as no restriction is put on the value of f at $x_0, \ldots, x_n$.

Exercise 1. What is the reasonable value of "area below graph of f" for step functions? Then show that the values of f at $x_0, \ldots, x_n$ do not matter as long as "area" is concerned.

Darboux integral

Definition 5. (Darboux sums and Darboux integrals)

- *For any fixed partition P, define the upper/lower sums as*

$$U(f, P) := \sum_{j=1}^{n} M_j(f)\,(x_j - x_{j-1}); \qquad L(f, P) := \sum_{j=1}^{n} m_j(f)\,(x_j - x_{j-1}) \tag{1}$$

  where $M_j(f) = \max_{[x_{j-1}, x_j]} f(x)$, $m_j(f) = \min_{[x_{j-1}, x_j]} f(x)$.

- *Now define the upper/lower integrals:*

$$U(f) := \inf_P U(f, P); \qquad L(f) := \sup_P L(f, P). \tag{2}$$

- *If $U(f) = L(f)$ then we say $f(x)$ is Darboux integrable on $[a, b]$, and denote this common value by $\int_a^b f(x)\,\mathrm{d}x$.*

Exercise 2. Find a function that is not Darboux integrable on $[0, 1]$.

Riemann integral

Definition 6. (Riemann sum and Riemann integral) *Let $f\colon [a, b] \mapsto \mathbb{R}$.*

- *For any partition P of $[a, b]$, and any $\xi_1, \ldots, \xi_n$ satisfying*

$$\xi_i \in [x_{i-1}, x_i] \tag{3}$$

we define the Riemann sum:

$$I(f, P, \xi_1, \ldots, \xi_n) := \sum_{i=1}^{n} f(\xi_i)\,(x_i - x_{i-1}). \tag{4}$$

- *Then if there is $A \in \mathbb{R}$ such that*

$$\lim_{\max\{|x_i - x_{i-1}|\} \longrightarrow 0} I(f, P, \xi_1, \ldots, \xi_n) = A \tag{5}$$

we say f is Riemann integrable on $[a, b]$ and denote the limit A by $\int_a^b f(x)\,\mathrm{d}x$.

Exercise 3. The limit in the above definition is not our usual limit. Explain how should it be understood.

Exercise 4. Show that your function in Exercise 2 is not Riemann integrable either.

Equivalence of Darboux and Riemann integrals

Remark 7. The Darboux upper and lower integrals are the best values for "area below graph of f" through approximation using step functions form above and below. The idea of Riemann is: If whatever approximation I use all eventually lead to the same value, then this value has to be the reasonable "area below graph of f".

Theorem 8. *$f: [a, b] \mapsto \mathbb{R}$ is Riemann integrable if and only if it is Darboux integrable.*

Exercise 5. Prove the theorem.

Integrability of functions

Consider $f: [a, b] \mapsto \mathbb{R}$.

- f is integrable $\Longrightarrow$ f is bounded. The converse does not hold.
- f is continuous on $[a, b] \Longrightarrow f$ is integrable. The converse does not hold.
- f is continuous on $[a, b]$ except at finitely many points $\Longrightarrow$ f is integrable. The converse does not hold.
- f is monotone on $[a, b]$ $\Longrightarrow$ f is integrable. The converse does not hold.

Exercise 6. Justify (through proof or counterexample) the above statements.

Exercise 7. Find a monotone function on $[0, 1]$ that has infinitely many discontinuous points.

Exercise 8. Prove that the Riemann function

$$f(x) := \begin{cases} \frac{1}{q} & x = \frac{p}{q},\ \ p, q \text{ coprime} \\ 0 & x \notin \mathbb{Q} \end{cases} \tag{6}$$

is integrable on $[0, 1]$.

Understanding integration

Exercise 9. Consider the following approach to establish an integration theory for f: $[a, b] \mapsto \mathbb{R}$. For any partition $P = \{x_0, \ldots, x_n\}$, define the T-sum[1]:

$$T(f, P) := \sum_{i=1}^{n} f\Big(\frac{x_i + x_{i-1}}{2}\Big)\,(x_i - x_{i-1}). \tag{7}$$

We say f is T-integrable if the following limit exists:

$$\lim_{\max\{|x_i - x_{i-1}|\} \longrightarrow 0} T(f, P). \tag{8}$$

What are the relations between T-integrals and Riemann/Darboux integrals? Justify your answers.

- An "integral" is a function $I: X \mapsto \mathbb{R}$ where X is the set of all pairs (A, f) with $A \subseteq \mathbb{R}$ and $f: \mathbb{R} \mapsto \mathbb{R}$ satisfying certain properties.

  Exercise 10. List the properties you think are reasonable and check whether Riemann integral satisfies them.

- All the functions f for which $I(A, f)$ can be defined are said to be "$\cdots$-integrable" on A.

- All the sets A for which $I(A, 1)$ can be defined are said to be "$\cdots$-measurable".

Exercise 11. Apply the above understanding to argue that the definition of improper integrals are reasonable.

Exercise 12. Let $E \subseteq \mathbb{R}$ be an arbitrary subset. Try to give a reasonable definition for $\int_E f(x)\, \mathrm{d}x$. What are your criteria for "reasonable"? What new difficulty do you encounter? Any idea to overcome it?

Remark 9. Those interested in deeper understanding of integration should check out Chapter 1 of T. Claesson and L. Hormander's beautiful little note *Integrationsteori* (Studentlitteratur 1970).

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1. T stands for trapezoidal. Note that this is not standard terminology.