# Taylor expansion

## Taylor polynomial (expansion with Peano form of the remainder)

**Definition 1.** *Let $f(x)$ be kth differentiable on $(a,b)$ for $k=1,2,\ldots,n-1$ and $f^{(n)}(x_0)$ exists for $x_0\in(a,b)$. Then the polynomial*

$$P_n(x):=f(x_0)+f'(x_0)\,(x-x_0)+\frac{f''(x_0)}{2}\,(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}\,(x-x_0)^n \tag{1}$$

*is called the nth degree Taylor polynomial of $f(x)$ at $x_0$.*

**Exercise 1.** Is it enough to assume only the existence of $f'(x_0),\ldots,f^{(n)}(x_0)$?

**Exercise 2. (What is special about $P_n$?)** Prove that for any other $n$th degree polynomial $Q_n(x)$,

$$\lim_{x\longrightarrow x_0}\frac{f(x)-P_n(x)}{f(x)-Q_n(x)}=0. \tag{2}$$

The difference $R_n(x):=f(x)-P_n(x)$ is called the "remainder". This is equivalent to

$$\lim_{x\longrightarrow x_0}\frac{R_n(x)}{(x-x_0)^n}=0. \tag{3}$$

**Exercise 3.** Prove that (3) and (2) are equivalent.

(Hint: We need to show

$$\lim\frac{R_n(x)}{R_n(x)+Q_n(x)}=0 \text{ for every } n\text{th degree polynomial } Q_n\Longleftrightarrow\lim\frac{R_n(x)}{(x-x_0)^n}=0. \tag{4}$$

$\Longleftarrow$ is relatively easy: Let $(x-x_0)^k$ be the lowest order term of $Q_n$. We simply divide both numerator and denominator by $(x-x_0)^k$ and then take limit. For $\Longrightarrow$, take $Q_n(x)=(x-x_0)^n$, and try to prove the conclusion using definition ($\varepsilon$-$\delta$ stuff).

**Exercise 4.** Let $n\in\mathbb{N}$. Let $f(x)$ be such that there exists a $n$th degree polynomial $Q_n(x)$ satisfying

$$f(x)=Q_n(x)+R_n(x) \tag{5}$$

with $R_n(x)/(x-x_0)^n\longrightarrow 0$ as $x\longrightarrow x_0$. Can we conclude the existence of $f^{(k)}(x_0)$, $k=1,2,\ldots,n$? Can we conclude the existence of $\delta>0$ such that $f^{(k)}(x)$ exists for $k=1,2,\ldots,n-1$ for all $x\in(x_0-\delta,x_0+\delta)$? (Hint: Take your nowhere continuous function and multiply by $(x-x_0)^{n+1}$)

**Remark 2.** Note that all the above is obtained only assuming the existence of $f^{(n)}(x_0)$ at one single point $x_0$, that is the differentiability of $f^{(n-1)}(x)$ at one single point. If we assume more, we will be able to obtain more precise formulas for the remainder $R_n(x)$.

## Taylor expansion with Lagrange form (and other forms) of the remainder

**Theorem 3. (Lagrange form of the remainder)** *Let $f^{(k)}(x)$ be continuous on $[a,b]$ for all $k=1,2,\ldots,n$. Let $f^{(n+1)}(x)$ exist on $(a,b)$. Then there is $\xi\in(a,b)$ such that*

$$R_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}\,(x-x_0)^{n+1}. \tag{6}$$

**Remark 4.** This gives us more information than $R_n(x)/(x-x_0)^n\longrightarrow 0$.

**Remark 5.** It should be clear that $\xi$ depends on $x$.

**Exercise 5.** Prove the theorem as follows. Fix $x, x_0$. Define

$$F(t) = f(t) - \left[ f(x_0) + f'(x_0)\,(t - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}\,(t - x_0)^n \right]; \qquad G(t) = (t - x_0)^{n+1}. \tag{7}$$

Apply Cauchy's MVT $n$ times to $\frac{F(t) - F(x_0)}{G(t) - G(x_0)}$.

**Exercise 6.** Prove the theorem as follows. Fix $x, x_0$. Define

$$F(t) = f(x) - \left[ f(t) + f'(t)\,(x - t) + \cdots + \frac{f^{(n)}(t)}{n!}\,(x - t)^n \right]; \qquad G(t) = (x - t)^{n+1}. \tag{8}$$

Then apply Cauchy's MVT to $\frac{F(x_0) - F(x)}{G(x_0) - G(x)}$.

**Exercise 7. (Cauchy form of the remainder)** Taking $G(t) = x - t$ to prove the following Cauchy form of the remainder:

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{n!}\,(x - \xi)^n\,(x - x_0) = \frac{f^{(n+1)}(x_0 + \theta\,(x - x_0))}{n!}\,(1 - \theta)^n\,(x - x_0)^{n+1} \tag{9}$$

where $\theta = \frac{\xi - x_0}{x - x_0} \in (0, 1)$.

**Exercise 8. (Schlomilich-Roche form of the remainder)** Taking $G(t) = (x - t)^p$ to prove

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{n!\,p}\,(x - \xi)^{n-p+1}\,(x - x_0)^p = \frac{f^{(n+1)}(x_0 + \theta\,(x - x_0))}{n!\,p}\,(1 - \theta)^{n+1-p}\,(x - x_0)^{n+1} \tag{10}$$

where $\theta = \frac{\xi - x_0}{x - x_0} \in (0, 1)$.

**Example 6.** Estimate the remainder of the expansion of $\ln(1 + x)$ at $x = 0$ to $n$th degree for $x \in (-1, 1)$.

**Solution.** If we write the remainder in its Lagrange form:

$$|R_n| = \frac{1}{n+1} \left| \frac{x}{\xi} \right|^{n+1} \tag{11}$$

For which we have $\xi$ between 1 and $1 + x$. Thus when $x > -1/2$, we can get a good estimate to show $R_n \longrightarrow 0$; On the other hand when $x < -1/2$ this strategy won't work.

In this case we can use the Cauchy form:

$$|R_n| = \frac{1}{|1 + \theta\,x|^{n+1}}\,(1 - \theta)^n\,x^{n+1}. \tag{12}$$

**Exercise 9.** Fill in all the details for the above solution.

**Exercise 10.** What do you think is the reason that we restrict our consideration to $x \in (-1, 1)$?

### Taylor expansion with integral form of the remainder

**Theorem 7. (Integral form of the remainder)** *Assume $f^{(n+1)}(t)$ is integrable on $(a, b)$, $x_0, x \in (a, b)$. Then*

$$R_n(x) = \frac{1}{n!} \int_{x_0}^{x} f^{(n+1)}(t)\,(x - t)^n\,\mathrm{d}t. \tag{13}$$

**Exercise 11.** Prove the above theorem using mathematical induction and integration by parts.

**Remark 8.** The advantage of the integral form of remainder over all previous types of remainder is that everything involved: $f^{(n+1)}, (x-t)^n$ are differentiable and thus can be subject to further operations. On the other hand, the dependence of $\xi$ on $x$ is quite mysterious, there is even no guarantee that $\xi(x)$ is continuous. However, see the following exercise.

**Exercise 12.** Let $f:(a,b)\mapsto\mathbb{R}$ and $x_0\in(a,b)$. Assume that $f^{(n+2)}(x)$ exists and is continuous on $(a,b)$ with $f^{(n+1)}(x_0)\neq 0$. Let

$$f(x)=P_n(x)+\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1} \tag{14}$$

be the Taylor expansion. Define $\xi(x_0)=x_0$. Prove that $\xi$ is differentiable at $x_0$ and $\xi'(x)=1/(n+2)$.