Continuity and differentiability

Continuity

Definition 1. *A function $f(x):\mathbb{R}\mapsto\mathbb{R}$ is continuous at $x_0\in\mathbb{R}$ if and only if $\lim_{x\longrightarrow x_0}f(x)$ exists and equals $f(x_0)$.*

Exercise 1. Give an example of $f(x)$ and x_0 such that $\lim_{x\longrightarrow x_0}f(x)$ exists but does not equal $f(x_0)$. Such discontinuity is called *discontinuity of the first kind*. Explain why it is reasonable to write

$$\frac{x^2-9}{x-3}=x+3. \tag{1}$$

Exercise 2. Give an example of $f(x)$ and x_0 such that $\lim_{x\longrightarrow x_0}f(x)$ does not exist. Such discontinuity is called *discontinuity of the second kind*.

Exercise 3. Give an example of $f(x)$ on $[a,b]$ that is discontinuous at every $x\in[a,b]$.

Exercise 4. Give an example of $f(x)$ on $[a,b]$ that is continuous at every $x\notin\mathbb{Q}$ but discontinuous at every $x\in\mathbb{Q}$.

Exercise 5. Let $f:\mathbb{R}\mapsto\mathbb{R}$ be a function. Prove that

$$f \text{ is continuous everywhere}\Longleftrightarrow\forall\text{open set } A\subseteq\mathbb{R},\quad f^{-1}(A) \text{ is open.} \tag{2}$$

Here the preimage $f^{-1}(A):=\{x\in\mathbb{R}|\ f(x)\in A\}$.

Now let $x_0\in\mathbb{R}$. Prove that

$$f \text{ is continuous at } x_0\Longleftrightarrow\forall \text{ open set } A\subseteq\mathbb{R},\ f(x_0)\in A,\ \text{there is open set } V \text{ such that } x_0\in V\subseteq f^{-1}(A). \tag{3}$$

Give an example to show that $f^{-1}(A)$ may not be open.

Definition 2. *A function $f(x)$: $[a,b]\mapsto\mathbb{R}$ is said to be continuous at a if $\lim_{x\longrightarrow a+}f(x)=f(a)$; Continuity at b is defined similarly.*

Exercise 6. How to understand the above definition through the idea of topology introduced earlier?

Theorem 3. (Intermediate value theorem) *Let $a,b\in\mathbb{R}$ and $f(x)$ be continuous on $[a,b]$. Then for any $s\in(\min(f(a),f(b)),\max(f(a),f(b)))$, there is $\xi\in(a,b)$ such that $f(\xi)=s$.*

Exercise 7. Would continuity of f on (a,b) suffice? Why?

Exercise 8. Let $f:[a,b]\mapsto\mathbb{R}$ be continuous and one-to-one. Prove that $f(x)$ is monotone and therefore its inverse function exists. Then prove that the inverse function is continuous too.

Exercise 9. Let $f:[a,b]\mapsto\mathbb{R}$ be such that for any $s\in(\min(f(a),f(b)),\max(f(a),f(b)))$, there is $\xi\in(a,b)$ such that $f(\xi)=s$. Must f be continuous? Justify your answer.

Exercise 10. Is there any reasonable generalization of the theorem to the cases $a=-\infty$ or $b=+\infty$? Justify.

Theorem 4. (Extreme values) *Let $a,b\in\mathbb{R}$, and $f:[a,b]\mapsto\mathbb{R}$ be continuous. Then there are $x_{\max}$, $x_{\min}\in[a,b]$ such that for all $x\in[a,b]$, $f(x_{\max})\geqslant f(x)\geqslant f(x_{\min})$.*

Theorem 5. (Uniform continuity) *Let $f: [a,b] \mapsto \mathbb{R}$ with a, b finite be continuous. Then for any $\varepsilon > 0$ there is $\delta > 0$ such that $\forall x, y \in [a,b], |x - y| < \delta \Longrightarrow |f(x) - f(y)| < \varepsilon$.*

Exercise 11. Can we replace $[a,b]$ by (a,b) or $(a,b]$? Can $a = -\infty$ or $b = +\infty$?

Differentiability

Definition 6. *Let f: $(a, b) \mapsto \mathbb{R}$. Then $f(x)$ is differentiable at $x_0 \in (a, b)$ if and only if $\lim_{x \longrightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists and is finite.*

Exercise 12. How can we define differentiability at a or b if $f(x)$ is only defined on $[a,b]$?

Exercise 13. Let $f(x)$ be differentiable on (a,b). Can we draw any conclusion on continuity of f at a or b?

Exercise 14. Find the mistake in the following proof of chain rule:

$$\begin{aligned}\lim_{x \longrightarrow x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} &= \lim_{x \longrightarrow x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \frac{f(x) - f(x_0)}{x - x_0} \\ &= \left(\lim_{f(x) \rightarrow f(x_0)} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \frac{f(x) - f(x_0)}{x - x_0} \right) \left(\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \right) \\ &= g'(f(x_0))\, f'(x_0). \end{aligned} \tag{4}$$

Now read any single variable analysis textbook and understand how this is overcome.

Theorem 7. (Derivatives from approximation point of view) *Let $f(x)$: $(a, b) \mapsto \mathbb{R}$ and $x_0 \in (a,b)$. Then $f(x)$ is differentiable at x_0 if and only if there is a linear function $g(x) = k\,(x - x_0)$ for some $k \in \mathbb{R}$ such that*

$$\lim_{x \longrightarrow x_0} \frac{f(x) - f(x_0) - g(x)}{x - x_0} = 0. \tag{5}$$

Exercise 15. Let $f(x)$ be differentiable at x_0. Prove that

$$\lim_{x \longrightarrow x_0} \frac{|f(x) - f(x_0) - f'(x_0)\,(x - x_0)|}{|f(x) - b - a\,(x - x_0)|} = 0 \tag{6}$$

unless $b = f(x_0)$ and $a = f'(x_0)$. Thus $f(x_0) + f'(x_0)\,(x - x_0)$ is the best first order approximation of $f(x)$ near x_0. (Hint: You need to apply L'Hospital)

Exercise 16. Prove the converse. If there is a best first order approximation of f at x_0, must f be differentiable there? Justify your answer.

Theorem 8. (Mean value theorems) *Let $f(x)$, $g(x)$ be continuous on $[a, b]$ and differentiable on (a,b). Further assume that $g'(x) \neq 0$ on (a,b).*

a) **(Lagrange MVT)** *There is $\xi \in (a,b)$ such that*

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}; \tag{7}$$

b) **(Cauchy MVT)** *There is* $\xi \in (a, b)$ *such that*

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)}. \tag{8}$$

Exercise 17. Let $f: (a, b) \mapsto \mathbb{R}$ be differentiable. Prove that if $f'(x)$ is discontinuous at $x_0 \in (a, b)$, then the discontinuity is of the second kind. In other words, if both $\lim_{x \longrightarrow x_0+} f'(x)$ and $\lim_{x \longrightarrow x_0-} f'(x)$ exist, then they must be equal and $f'(x)$ is continuous at x_0.

Exercise 18. (Darboux Theorem) Let $f: (a, b) \mapsto \mathbb{R}$ be differentiable. Then $f'(x)$ satisfies the intermediate value property on any $[c, d] \subset (a, b)$.

Theorem 9. (Max and Min) *Let* $f(x)$ *be continuous and differentiable on* $[a, b]$*. Let* x_0 *be the maximizer (or minimizer) of* f *over* $[a, b]$*. Then*

a) $x_0 \in (a, b) \Longrightarrow f'(x_0) = 0$;

b) $x_0 = a \Longrightarrow f'(a) \leqslant 0 (\geqslant 0)$;

c) $x_0 = b \Longrightarrow f'(b) \geqslant 0 (\leqslant 0)$.

Exercise 19. Discuss the procedure of finding maximum of a differentiable $f(x)$.

Exercise 20. Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b). Let a be a maximizer that is $f(a) \geqslant f(x)$ for all $x \in [a, b]$. Can we conclude $\lim_{x \longrightarrow a} f'(x) \leqslant 0$? Justify.