1. Understanding limit of sequences.

1.1. Existence of limit.

Two common methods of establishing such existence are through Cauchy criterion and monotonicity.

Cauchy criterion.

Definition 1. *A sequence* $\{x_n\}$ *is said to be a Cauchy sequence if for any* $\varepsilon > 0$*, there is* $N \in \mathbb{N}$ *such that for any* $m, n > N$*,* $|x_m - x_n| < \varepsilon$*.*

Exercise 1. How would you define "Cauchy sequence" in a general topological space (X, Σ)? What sequences are Cauchy if Σ is the discrete or trivial topology?

Theorem 2. (Cauchy criterion) *Let* $\{x_n\}$ *be a sequence of real numbers. Then*

$$\{x_n\} \text{ is Cauchy} \Longleftrightarrow \lim_{n \longrightarrow \infty} x_n \text{ exists.} \tag{1}$$

Exercise 2. Can we replace $|x_m - x_n| < \varepsilon$ in the definition of Cauchy sequence by $|x_{n+1} - x_n| < \varepsilon$? Why?

Exercise 3. Design similar criterions for $\lim_{x \longrightarrow a} f(x)$ where $a \in \mathbb{R}$ or $a = \pm\infty$.

Remark 3. Implicit in the above theorem is that $\lim_{n \longrightarrow \infty} x_n$ exists and is still in $\mathbb{R}$.

Exercise 4. Let $\{x_n\} \subset \mathbb{Q}$ be a sequence of rational numbers. Can we define "Cauchy" for such sequences?

Monotonicity.

Theorem 4. *Let* $\{x_n\}$ *be a sequence of real numbers. Then* $\lim_{n \longrightarrow \infty} x_n$ *exists if either one of the following holds*

- $\{x_n\}$ *increases and has a finite upper bound;*
- $\{x_n\}$ *decreases and has a finite lower bound.*

Exercise 5. Let $\{x_n\} \subseteq \mathbb{Q}$ be increasing with finite upper bound. Can we conclude $\lim_{n \longrightarrow \infty} x_n$ exists in $\mathbb{Q}$? What is the crucial difference between $\mathbb{R}$ and $\mathbb{Q}$?

Exercise 6. For those who knows complex numbers: Let $\{x_n\} \subset \mathbb{C}$ be a sequence of complex numbers. Can we formulate the monotone convergence theorem for it? Can we formulate Cauchy criterion for it?

Example 5. Prove the existence of limit $\lim_{n \longrightarrow \infty} \left(1 + \frac{1}{n}\right)^n$ by showing

a) The sequence is increasing;

b) The sequence is bounded from above.

Proof.

a) We expand

$$\begin{aligned}\left(1 + \frac{1}{n}\right)^n &= 1 + \frac{n}{1}\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \cdots + \frac{n!}{n!}\frac{1}{n^n} \\ &= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \cdots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right);\end{aligned} \tag{2}$$

$$\begin{aligned}\left(1 + \frac{1}{n+1}\right)^{n+1} &= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n+1}\right) + \cdots + \frac{1}{n!}\left(1 - \frac{1}{n+1}\right)\cdots\left(1 - \frac{n-1}{n+1}\right) \\ &\quad + \frac{1}{(n+1)!}\left(1 - \frac{1}{n+1}\right)\cdots\left(1 - \frac{n}{n+1}\right).\end{aligned} \tag{3}$$

The relation is now obvious.

b) We have

$$\left(1+\frac{1}{n}\right)^n < 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!} < 1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}} < 3. \tag{4}$$

Thus ends the proof. □

Exercise 7. Using the above, prove

$$\lim_{x\longrightarrow\infty}\left(1+\frac{1}{x}\right)^x = \lim_{n\longrightarrow\infty}\left(1+\frac{1}{n}\right)^n. \tag{5}$$

1.2. Subsequences.

Definition 6. *Let $\{x_n\}$ be a sequence of real numbers. A subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is a composite function:*

$$x_{n_k} = f(g(k)) \tag{6}$$

where $f\colon \mathbb{N} \mapsto \mathbb{R}$ is defined by $f(n) := x_n$ and $g\colon \mathbb{N} \mapsto \mathbb{N}$ is defined by $g(k) := n_k$. We further require g to be strictly increasing, that is $k < l \Longrightarrow g(k) < g(l)$.[1]

Remark 7. It is important to understand that the "n" in the notation $\{x_{n_k}\}$ does not have any numerical value, it is k that is changing.

Theorem 8. *Let $\{x_n\}$ be a sequence of real numbers.*

a) If $\lim_{n\longrightarrow\infty} x_n = L \in \mathbb{R}$, then any subsequence $\{x_{n_k}\}$ satisfies $\lim_{k\longrightarrow\infty} x_{n_k} = L$.

b) If all subsequences $\{x_{n_k}\}$ converges to L, then $\lim_{n\longrightarrow\infty} x_n$ exists and equals L.

Exercise 8. Let $\{x_n\}$ be a real sequence. Assume that every subsequence of $\{x_n\}$ converges. Prove that $\{x_n\}$ converges.

1.3. Bolzano-Weierstrass.

Theorem 9. (Bolzano-Weierstrass) *Let $\{x_n\} \subset [a,b]$. Then there is a converging subsequence.*

Remark 10. In other words, every bounded sequence has a converging subsequence.

Exercise 9. Let $\{x_n\}$ be a bounded real sequence. Assume that every convergent subsequence has the same limit L, then $\lim_{n\longrightarrow\infty} x_n$ exists and equals L.

1.4. Liminf and limsup.

Definition 11. (liminf and limsup) *Let $\{x_n\}$ be a sequence of real numbers, we define*

$$\liminf_{n\longrightarrow\infty} x_n = \lim_{n\longrightarrow\infty} (\inf\{x_n, x_{n+1}, \ldots\}); \qquad \limsup_{n\longrightarrow\infty} x_n = \lim_{n\longrightarrow\infty} (\sup\{x_n, x_{n+1}, \ldots\}). \tag{7}$$

Exercise 10. Let $\{x_n\}$ be a sequence of real numbers. Let $A := \{L \in \mathbb{R} | \ \exists \text{subsequence } x_{n_k} \longrightarrow L\}$. Then

$$\liminf_{n\longrightarrow\infty} x_n = \inf A; \qquad \limsup_{n\longrightarrow\infty} x_n = \sup A. \tag{8}$$

Further prove that $\lim_{n\longrightarrow\infty} x_n = L \Longleftrightarrow \liminf_{n\longrightarrow\infty} x_n = \limsup_{n\longrightarrow\infty} x_n = L$.

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1. *Note that as a consequence we always have $n_k \geqslant k$.*