Limit of sequences and functions

Understand the definitions

Let $f(x)\colon \mathbb{R} \mapsto \mathbb{R}$ be a function, then we have the following definitions for its limiting behavior:

- $\lim_{x \longrightarrow a} f(x) = L \Longleftrightarrow \forall \varepsilon > 0\ \exists \delta > 0$ such that
$$0 < |x - a| < \delta \Longrightarrow |f(x) - L| < \varepsilon. \tag{1}$$
- $\lim_{x \longrightarrow a} f(x) = \infty \Longleftrightarrow \forall M \in \mathbb{R}\ \exists \delta > 0$ such that
$$0 < |x - a| < \delta \Longrightarrow f(x) > M. \tag{2}$$
- $\lim_{x \longrightarrow -\infty} f(x) = L \Longleftrightarrow \forall \varepsilon > 0\ \exists M \in \mathbb{R}$ such that
$$x < M \Longrightarrow |f(x) - L| < \varepsilon. \tag{3}$$
- ...

Exercise 1. Let $f(x)\colon \mathbb{R} \mapsto \mathbb{R}$ be a function satisfying any one of the following. What can we conclude about f?

a) $\exists \delta > 0,\ \forall \varepsilon > 0,\ 0 < |x - a| < \delta \Longrightarrow |f(x) - L| < \varepsilon;$

b) $\exists \delta > 0,\ \forall M \in \mathbb{R}, 0 < |x - a| < \delta \Longrightarrow f(x) > M;$

c) $\exists M > 0,\ \forall \varepsilon > 0, \quad x < M \Longrightarrow |f(x) - L| < \varepsilon.$

We further define one-sided limits:

- $\lim_{x \longrightarrow a+} f(x) = L \Longleftrightarrow \forall \varepsilon > 0\ \exists \delta > 0$ such that
$$0 < x - a < \delta \Longrightarrow |f(x) - L| < \varepsilon. \tag{4}$$
- $\lim_{x \longrightarrow a-} f(x) = -\infty \Longleftrightarrow \forall M \in \mathbb{R}\ \exists \delta > 0$ such that
$$-\delta < x - a < 0 \Longrightarrow f(x) < M. \tag{5}$$
- ...

There are also definitions for a sequence of real numbers $\{x_n\}$:

- $\lim_{n \longrightarrow \infty} x_n = L \Longleftrightarrow \forall \varepsilon > 0\ \exists N \in \mathbb{N}$ such that
$$n > N \Longrightarrow |x_n - L| < \varepsilon. \tag{6}$$
- $\lim_{n \longrightarrow \infty} x_n = \infty \Longleftrightarrow \forall M \in \mathbb{R}\ \exists N \in \mathbb{N}$ such that
$$n > N \Longrightarrow x_n > M. \tag{7}$$
- ...

These definitions all look different, yet at the same time similar. Indeed, the intuition of “limit” is simply

> As one quantity approaches a certain value, a related quantity approaches another certain value.

The question now is, can we understand them in a unified framework? The answer is yes. All we need to do is to give precise meaning to “approach” in various situations. In fact, given any set – not necessarily $\mathbb{R}$ or $\mathbb{N}$ – we can give precise meaning to something “approaching an element of the set” through defining a so-called “topological structure” on this set.

Topology

Definition 1. (Topological structure) *Let X be a set. A topological structure on X is the collection Σ of subsets of X satisfying*

i. $X \in \Sigma$; $\varnothing \in \Sigma$;

ii. The union of any number of elements in Σ is an element of Σ;

iii. The intersection of finitely many elements in Σ is an element of Σ.

The elements in Σ are called “open sets” in this particular topology (X, Σ).

Remark 2. It should be emphasized that a topology is a pair $(X,\ \Sigma)$. Different topological structures can be put on the same set X to form different topologies.

Exercise 2. Let

$$\Sigma_1 = \{\text{open sets} \subseteq \mathbb{R}\}; \tag{8}$$

$$\Sigma_2 = \{\text{closed sets} \subseteq \mathbb{R}\}; \tag{9}$$

$$\Sigma_3 = \{\text{half-open-half-closed intervals } [a, b) \subseteq \mathbb{R}\} \tag{10}$$

$$\Sigma_4 = \{A |\ A \subseteq \mathbb{R}\} \tag{11}$$

$$\Sigma_5 = \{\varnothing, \mathbb{R}\}. \tag{12}$$

Consider $(\mathbb{R}, \Sigma_i)$, $i = 1, 2, 3, 4, 5$. Which pairs are topologies and which are not? Justify your answers.

Exercise 3. Let X be any set. Let $\Sigma_1 = \{\varnothing, X\}$, $\Sigma_2 = \{A |\ A \subseteq X\}$. Prove that $(X, \Sigma_1), (X, \Sigma_2)$ are topologies.

Remark 3. The two topologies in the above exercise are called “trivial topology” and “discrete topology” respectively. The usual topology on $\mathbb{R}$ is called its “natural topology”.

Definition 4. (Limit) *Let (X_1, Σ_1) and (X_2, Σ_2) be two topologies. Let $f: X_1 \mapsto X_2$. Let $a \in X_1$ and $L \in X_2$. Then we say $\lim_{x \longrightarrow a} f(x) = L$ if for every $B \in \Sigma_2$ such that $L \in B$, there is $A \in \Sigma_1$ satisfying $a \in A$ and $f(A - \{a\}) \subseteq B$, that is, for any $x \in A, x \neq a$, $f(x) \in B$.*

Exercise 4. Let $f\colon\mathbb{R}\mapsto\mathbb{R}$. Let $a, L\in\mathbb{R}$. Recall that we define $\lim_{x\longrightarrow a} f(x) = L \Longleftrightarrow \forall\varepsilon>0\ \exists\delta>0$ such that

$$0<|x-a|<\delta \Longrightarrow |f(x)-L|<\varepsilon. \tag{13}$$

What is the topology we assign to $\mathbb{R}$ here?

Exercise 5. Let $f\colon\mathbb{R}\mapsto\mathbb{R}$ be an arbitrary function and $a\in\mathbb{R}$ be arbitrary. We assign the domain $\mathbb{R}$ with the open interval topology and the range $\mathbb{R}$ with the trivial topology. Prove that $\lim_{x\longrightarrow a} f(x) = L$ for any $L\in\mathbb{R}$.

Remark 5. The above obviously violated a fundamental theorem – If limit exists then it is unique – that we learned in single variable calculus. The reason is that, the trivial topology cannot tell different points apart. In topological terminology, $\mathbb{R}$ with trivial topology is not "Hausdorff". For a quick tour of topology, see Chapter 1 of *Manifolds, Tensor Analysis, and Applications* by Ralph Abraham, Jerrold E. Marsden and Tudor Ratiu.

Exercise 6. Study limit of $f\colon\mathbb{R}\mapsto\mathbb{R}$ when the domain-range topologies are of the other eight possibilities (each can be assigned the natural (open-interval), trivial, discrete topology).

Exercise 7. We know that a sequence $\{x_n\}$ can be identified with a function $f\colon\mathbb{N}\mapsto\mathbb{R}$ through $f(n) := x_n$. What topologies on $\mathbb{N}\cup\{\infty\}$ and $\mathbb{R}$ would give us the usual definition of $\lim_{n\longrightarrow\infty} x_n = L$?

Inherited topology

Theorem 6. (Inherited topology) *Let (X,Σ) be a topology. Let $Y\subset X$. Define $\Sigma_1 = \{A\cap Y | A\in\Sigma\}$. Then (Y,Σ_1) is a topology.*

Exercise 8. Prove the theorem.

Exercise 9. Prove that the inherited topology on $\mathbb{N}$ as subset of $\mathbb{R}$ with natural topology is its discrete topology.

Exercise 10. Consider the subset $(-\infty, a]\subset\mathbb{R}$. Equip $\mathbb{R}$ with its natural topology. What is the inherited topology? What kind of limit results from this topology, at the point a?

Exercise 11. Let $E\subseteq\mathbb{R}$. Let $\mathbb{R}$ be equipped with its natural topology. Let E be equipped with the inherited topology. Consider $f\colon E\mapsto\mathbb{R}$. For what kind of points $a\in E$ can we define the existence of $\lim_{x\longrightarrow a} f(x)$?

Topology on $\mathbb{R}^N$

Think: What is the natural topology on $\mathbb{R}^N$?