Math 117 Fall 2014 Midterm 3 Review Problems

- Midterm 3 coverage:
  - Lectures 26, 29 - 36 and the exercises therein. (Note that Integration is not included)
  - Required sections in Dr. Bowman's book and my 314 notes.
  - Homeworks 6 - 8.
  - The exercises below are to help you on the concepts and techniques. The exam problems may or may not look like them.
- Exercises.

Exercise 1. Prove that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent. (Sol:[1])

Exercise 2. Prove by ε-δ: $\text{sign}(x) := \begin{cases} 1 & x>0 \\ 0 & x=0 \\ -1 & x<0 \end{cases}$ is continuous at $a \neq 0$ but discontinuous at $a=0$. (Sol:[2])

Exercise 3. Prove by ε-δ: $f(x) = \begin{cases} 1/x^2 & x\neq 0 \\ 0 & x=0 \end{cases}$ is continuous at $a\neq 0$ and discontinuous at $a=0$. (Sol:[3])

Exercise 4. Prove or disprove: $f(x)$ is continuous at $a\in\mathbb{R}$ if and only if for every $\{x_n\}$ with $\lim_{n\to\infty} x_n = a$, there holds $\lim_{n\to\infty} f(x_n) = f(a)$. (Sol:[4])

Exercise 5. Let $f(x) := \begin{cases} x\sin\frac{1}{x} & x\neq 0 \\ 0 & x=0 \end{cases}$. Is $f(x)$ a continuous function (that is continuous at every a in its domain)? Justify your claim. (Sol:[5])

Exercise 6. Let $f(x) := \begin{cases} x + x^2\cos\frac{1}{x^4} & x\neq 0 \\ 0 & x=0 \end{cases}$. Prove that f is differentiable everywhere on $\mathbb{R}$ and calculate $f'(x)$. (Sol:[6])

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1. Since the series is positive, the partial sums $s_n := \sum_{k=1}^{n} \frac{1}{k^2}$ is increasing. All we need to show is that it has an upper bound. Let n be arbitrary. We have

$$\sum_{k=1}^{n} \frac{1}{k^2} < 1 + \sum_{k=2}^{n} \frac{1}{(k-1)\,k} = 2 - \frac{1}{n} < 2. \tag{1}$$

2. When $a \neq 0$, let $\varepsilon > 0$ be arbitrary. Set $\delta = |a|$. Then for all $|x-a| < \delta$ x has the same sign as a, consequently $|\text{sign}(x) - \text{sign}(a)| = 0 < \varepsilon$. When $a=0$, let $\delta > 0$ be arbitrary, take $x \in (0,\delta)$. Then we have $|\text{sign}(x) - \text{sign}(0)| = 1 \geqslant 1$.

3. Let $a\neq 0$. Let $\varepsilon > 0$ be arbitrary. Set $\delta < \min\left\{\frac{|a|}{2}, \frac{|a|^3\,\varepsilon}{8}\right\}$. Then for every $0<|x-a|<\delta$, we have

$$|f(x) - f(a)| = \left|\frac{1}{x^2} - \frac{1}{a^2}\right| = |x-a|\,\frac{|x+a|}{|x|^2\,|a|^2} < \delta\,\frac{2\,|a|}{|a|^4/4} = \delta\,\frac{8}{|a|^3} < \varepsilon. \tag{2}$$

At $a=0$, let $\delta>0$ be arbitrary. Set $x := \min\{1,\delta\}$. Then we have

$$|f(x) - f(a)| = \left|\frac{1}{x^2} - 0\right| = \frac{1}{|x|^2} \geqslant 1. \tag{3}$$

4. The claim is true. "If": We need to show $\lim_{x\to a} f(x) = f(a)$. We know that this is equivalent to for every $\{x_n\}$ satisfying $\lim_{n\to\infty} x_n = a, x_n \neq a$, there holds $\lim_{n\to\infty} f(x_n) = f(a)$. This obviously holds as the assumption says for every $\{x_n\}$ satisfying $\lim_{n\to\infty} x_n = a$, $\lim_{n\to\infty} f(x_n) = f(a)$. "Only if": Let $\{x_n\}$ be a arbitrary sequence with $\lim_{n\to\infty} x_n = a$. Let $\varepsilon > 0$ be arbitrary. As f is continuous at a, there is $\delta > 0$ such that when $|x-a|<\delta$, $|f(x) - f(a)| < \varepsilon$. Now as $\lim_{n\to\infty} x_n = a$, there is $N\in\mathbb{N}$ such that when $n \geqslant N$, $|x_n - a| < \delta$. Therefore when $n\geqslant N$, we have $|f(x_n) - f(a)| < \varepsilon$, that is $\lim_{n\to\infty} f(x_n) = f(a)$.

5. Yes. At $x\neq 0$ the continuity follows from the continuity of $x, \sin x$ and $1/x$. At $x=0$, we have by Squeeze $\lim_{x\to 0} f(x) = 0$.

6. As $x, x^2, \cos x$ are differentiable everywhere, and $\frac{1}{x^4}$ is differentiable at every $x\neq 0$, $f(x)$ is differentiable at every $x\neq 0$, and

$$f'(x) = 1 + 2\,x\cos\frac{1}{x^4} + 4\frac{1}{x^3}\sin\frac{1}{x^4}. \tag{4}$$

At 0, we have

$$\lim_{x\to 0}\frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0}\frac{x + x^2\cos\frac{1}{x^4}}{x} = \lim_{x\to 0}\left[1 + x\cos\frac{1}{x^4}\right] = 1 \tag{5}$$

so $f'(0) = 1$.

Exercise 7. Let $f(x) := \begin{cases} x + x\cos\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$. Prove that f is differentiable everywhere except at $x=0$. (Sol:[7])

- More exercises

Exercise 8. Prove that $\sum_{n=1}^{\infty} \frac{\cos n^2}{n^2}$ is convergent. (Sol:[8])

Exercise 9. Let $a_n > 0$. Prove the following:

- If $\limsup_{n\to\infty} a_n^{1/n} < 1$, then $\sum_{n=1}^{\infty} a_n$ converges;
- If $\limsup_{n\to\infty} a_n^{1/n} > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

(Sol:[9])

Exercise 10. Let $a_n > 0$ and $\sum_{n=1}^{\infty} a_n$ be convergent. Prove that $\sum_{n=1}^{\infty} a_n^2$ is convergent.

Exercise 11. Let $\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n, \sum_{n=1}^{\infty} c_n$ satisfy

i. $\forall n \in \mathbb{N}, a_n \leqslant b_n \leqslant c_n$;

ii. $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} c_n$ converge to the same sum.

Prove or disprove: $\forall n \in \mathbb{N}, a_n = b_n = c_n$.

Exercise 12. Let $f(x)$ be continuous at $a \in \mathbb{R}$. Let $g(x)$ be such that $\forall x \in \mathbb{R}, |g(x)| < |f(x)|$. Does it follow that $g(x)$ is continuous at a? Justify your claim. (Sol:[10])

Exercise 13. Prove by induction that all polynomials are continuous at every $a \in \mathbb{R}$.

Exercise 14. A function $f: A \mapsto \mathbb{R}$ $(A \subseteq \mathbb{R})$ is uniformly continuous on $B \subseteq A$ if and only if

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x, x' \in B, |x - x'| < \delta, \qquad |f(x) - f(x')| < \varepsilon. \tag{7}$$

a) Find a function $f: \mathbb{R} \mapsto \mathbb{R}$ that is continuous but not uniformly continuous on $\mathbb{R}$.

b) Write down the working negation of uniform continuity.

c) Prove that if $f: A \mapsto \mathbb{R}$ is continuous on $[a,b] \subseteq A$, then it is uniformly continuous on $[a,b]$. (Sol:[11])

Exercise 15. Prove that the equation $x^2 - 4\sin x + 1 = 0$ has at least two solutions in $\mathbb{R}$.

Exercise 16. Let $f: \mathbb{R} \mapsto \mathbb{R}$ be differentiable at $a \in \mathbb{R}$. Prove that $\lim_{h\to 0+} \frac{f(a+h) - f(a-h)}{2h} = f'(a)$. (Sol:[12])

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7. That $f(x)$ is differentiable whenever $x \neq 0$ is proved similarly to the previous problem. At 0, we have $\frac{f(x) - f(0)}{x - 0} = 1 + \cos\frac{1}{x}$ for $x \neq 0$. Now take $x_n = \frac{1}{2n\pi}$ and $y_n = \frac{1}{(2n+1)\pi}$. We have $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$, $x_n, y_n \neq 0$ but $\lim_{n\to\infty}\left(1 + \cos\frac{1}{x_n}\right) = 2$, $\lim_{n\to\infty}\left(1 + \cos\frac{1}{y_n}\right) = 0 \neq 2$. Therefore the limit $\lim_{x\to 0}\left(1 + \cos\frac{1}{x}\right)$ does not exist and consequently f is not differentiable at 0.

8. We prove it's Cauchy. Let $\varepsilon > 0$ be arbitrary. Set $N > \varepsilon^{-1}$. Then for every $m > n \geqslant N$, we have

$$\left|\sum_{k=n+1}^{m} \frac{\cos k^2}{k^2}\right| \leqslant \sum_{k=n+1}^{m} \frac{1}{k^2} < \sum_{k=n+1}^{m} \frac{1}{(k-1)k} = \sum_{k=n+1}^{m} \left(\frac{1}{k-1} - \frac{1}{k}\right) < \frac{1}{n} \leqslant \frac{1}{N} < \varepsilon. \tag{6}$$

9. We prove ii. Let $L := \limsup_{n\to\infty} a_n^{1/n} > 1$. Then by definition of limsup there holds $\sup\left\{a_n^{1/n}, a_{n+1}^{1/(n+1)}, \ldots\right\} \geqslant L$ for all $n \in \mathbb{N}$. By definition of sup we see that for every $n \in \mathbb{N}$, there is $k \geqslant n$ such that $a_k^{1/k} \geqslant 1$ and consequently $a_k \geqslant 1$. This implies $\lim_{n\to\infty} a_n = 0$ does not hold and the conclusion follows. For i try to prove that there is $r < 1$, $c > 0$ such that $a_n \leqslant c\,r^n$ for all $n \in \mathbb{N}$.

10. No. For example $f(x) \equiv 2$ and $g(x) = \operatorname{sign}(x)$.

11. Assume the contrary. Then there is $\varepsilon > 0$ such that for every $\delta > 0$ there are x_δ, y_δ satisfying $|x_\delta - y_\delta| < \delta$ but $|f(x_\delta) - f(y_\delta)| \geqslant \varepsilon$. Taking $\delta = 1/n$ for every $n \in \mathbb{N}$ we obtain two sequences $\{x_n\}$, $\{y_n\}$ such that $x_n - y_n \longrightarrow 0$ and $|f(x_n) - f(y_n)| \geqslant \varepsilon$. Thanks to Bolzano-Weierstrass there is a convergent subsequence $\{x_{n_k}\}$. Let $c := \lim_{k\to\infty} x_{n_k} \in [a,b]$. Then we have $\lim_{k\to\infty} f(x_{n_k}) = f(c)$. On the other hand as $x_n - y_n \to 0$ so does $x_{n_k} - y_{n_k}$ and consequently $\lim_{k\to\infty} y_{n_k} = c$ and $\lim_{k\to\infty} f(y_{n_k}) = f(c)$. But $|f(x_{n_k}) - f(y_{n_k})| \geqslant \varepsilon$ for all $k \in \mathbb{N}$. Contradiction.

12. Let $\varepsilon > 0$ be arbitrary. As $f'(a)$ exists, there is $\delta > 0$ such that for all $0 < |x - a| < \delta$, $\left|\frac{f(x) - f(a)}{x - a} - L\right| < \varepsilon$. Now let $0 < h < \delta$ be arbitrary. We see that $0 < |(a+h) - a| < \delta, 0 < |(a-h) - a| < \delta$. Consequently $\left|\frac{f(a+h) - f(a)}{h} - L\right|, \left|\frac{f(a-h) - a}{-h} - L\right| < \varepsilon$. By triangle inequality we see $\left|\frac{f(a+h) - f(a-h)}{2h} - L\right| = \frac{1}{2}\left|\left(\frac{f(a+h) - f(a)}{h} - L\right) + \left(\frac{f(a) - f(a-h)}{h} - L\right)\right| < \varepsilon$.

Exercise 17. Prove or disprove: Let $f: \mathbb{R} \mapsto \mathbb{R}$ and $a \in \mathbb{R}$ satisfy $\lim_{h \to 0+} \frac{f(a+h) - f(a-h)}{2h} = L \in \mathbb{R}$, then f is differentiable at a with $f'(a) = L$. (Hint:[13])

Exercise 18. Prove that $f(x) = 4x + x^3 + 2\sin x$ is strictly increasing on $\mathbb{R}$.

- Problems.

Problem 1. Let $\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n$ be convergent. Prove or disprove: $\sum_{n=1}^{\infty} \max\{a_n, b_n\}$ is convergent; $\sum_{n=1}^{\infty} \min\{a_n, b_n\}$ is convergent. (Hint:[14])

Problem 2. Prove or disprove: Let f: $\mathbb{R} \mapsto \mathbb{R}$ be continuous with $\limsup_{x \to +\infty} f(x) = A$, $\liminf_{x \to -\infty} f(x) = B$. Then for every s between A, B, there is $c \in \mathbb{R}$ such that $f(c) = s$. (Hint:[15])

Problem 3. Prove the Intermediate Value Theorem as follows:

a) Define $A := \{x | f(x) < s\}$, let $c_1 := \sup A$.

b) Define $B := \{x | f(x) > s\}$, let $c_2 := \inf B$.

In general does there hold $c_1 = c_2$? (Hint:[16])

Problem 4. Find a function $f(x)$ that is differentiable everywhere on $\mathbb{R}$ but $f'(x)$ is unbounded on $(-1, 1)$. Justify your claim.

Problem 5. How many solutions are there for $\ln(x+1) = x^2$? Justify your answer. (Hint:[17])

Problem 6. Find the exact number of solutions for $x^2 - 4\sin x + 1 = 0$. Justify your answer. (Hint:[18])

Problem 7. Prove

$$\forall x \in \left(-\frac{1}{2}, \frac{1}{2}\right), \qquad 3\arccos x - \arccos(3x - 4x^3) = \pi. \tag{8}$$

(Hint:[19])

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13. The claim is false. Consider $f(x) = |x|$.

14. Both are false. Consider $a_n = \frac{(-1)^n}{n}$, $b_n = \frac{(-1)^{n+1}}{n+1}$.

15. Apply IVT.

16. By definition of c_1 we have $f(x) \geqslant s$ for all $x > c_1$. If $c_1 > s$ there is $\delta > 0$ such that $f(x) > s$ for all $c_1 - \delta < x \leqslant c_1$. So $\sup A \leqslant c_1 - \delta$ contradiction. If $c_1 < s$ we show similarly $\sup A \geqslant c_1 + \delta$ for some $\delta > 0$.

17. 2. There is obviously no solution for $x < 0$. $x = 0$ is a solution. Let $f(x) := x^2 - \ln(1+x)$. We have $f'(x) = 2x - \frac{1}{1+x}$. We see that $f'(x) = 0$ has exactly one solution in $(0, \infty)$. Denote it by x_0. On $(0, x_0)$ $f'(x) < 0$ so f is strictly decreasing and on (x_0, ∞) $f(x)$ is strictly increasing.

18. To show that there are only two solutions, let $f(x) := x^2 - 4\sin x + 1$. First it is clear that there is no solution on $[-\pi, 0]$. For $x < -\pi$ we have $x^2 - 4\sin x + 1 \geqslant \pi^2 + 1 - 4 > 0$ so there is no solution on $(-\infty, -\pi)$ either. On $[0, \infty)$ we have $f'(x) = 2x - 4\cos x$ which has exactly one solution and therefore $f(x) = 0$ has exactly two solutions on $[0, \infty)$ and thus on $(-\infty, \infty)$.

19. Try to show derivative is 0.