Math 117 Fall 2014 Midterm 2 Review Problems

- Midterm 2 coverage:
  - Lectures 12 - 25 and the exercises therein.
  - Required sections in Dr. Bowman's book and my 314 notes.
  - Homeworks 3 - 5.
  - The exercises below are to help you on the concepts and techniques. The exam problems may or may not look like them.
  - Pages 7, 8, 10 of the Midterm Review of Math 314, 2013 may also help.
- Exercises.

Exercise 1. Prove the following **by definition**.

$$\lim_{n\to\infty} \frac{n^2+1}{n^3} = 0; \quad \lim_{n\to\infty} \frac{2^n}{n!} = 0; \quad \lim_{n\to\infty} \frac{n^3+1}{5\,n^2+7} = +\infty; \tag{1}$$

(Sol:[1])

Exercise 2. Prove the following **by definition.**

$$\lim_{x\to 1} \frac{x^2+1}{x^3+1} = 1; \quad \lim_{x\to 5-} \frac{x^2+1}{x-5} = -\infty; \quad \lim_{x\to 1} \sqrt{x+3} = 2. \tag{2}$$

(Sol:[2])

Exercise 3. Disprove the following **by definition**.

$$\lim_{n\to\infty} \frac{1}{n} = 2; \qquad \lim_{x\to 3} \frac{x^2+1}{x^2+x+5} = 2. \tag{3}$$

(Sol:[3];[4])

Exercise 4. Prove the following **by Squeeze**.

$$\lim_{n\to\infty} \frac{\sin n^3}{n^3} = 0; \qquad \lim_{n\to\infty} \frac{\sqrt{n^2-7}}{\sqrt{n^2+1}} = 1. \tag{4}$$

(Sol:[5];[6])

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1. Prove $\lim_{n\to\infty} \frac{n^2+1}{n^3} = 0$. Let $\varepsilon > 0$ be arbitrary. Take $N > \frac{2}{\varepsilon}$. Then for every $n \geqslant N$, we have $\left|\frac{n^2+1}{n^3} - 0\right| = \frac{n^2+1}{n^3} \leqslant \frac{n^2+n^2}{n^3} = \frac{2}{n} \leqslant \frac{2}{N} < \varepsilon$;

   Prove $\lim_{n\to\infty} \frac{2^n}{n!} = 0$. Let $\varepsilon > 0$ be arbitrary. Take $N \in \mathbb{N}$ such that $N > \frac{2}{\varepsilon}$. Then for every $n \geqslant N$, we have $\left|\frac{2^n}{n!} - 0\right| = \frac{2^n}{n!} = \frac{2}{n}\left(\frac{2^{n-1}}{(n-1)!}\right) \leqslant \frac{2}{n} \leqslant \frac{2}{N} < \varepsilon$.

   Prove $\lim_{n\to\infty} \frac{n^3+1}{5\,n^2+7} = +\infty$. Let $M > 0$ be arbitrary. Take $N > 12\,M$. Then for every $n \geqslant N$, we have $\frac{n^3+1}{5\,n^2+7} \geqslant \frac{n^3}{5\,n^2+7\,n^2} = \frac{n}{12} \geqslant \frac{N}{12} > M$.

2. Prove $\lim_{x\to 5-} \frac{x^2+1}{x-5} = -\infty$. Let $M < 0$ be arbitrary. Take $\delta = -\frac{1}{M}$. Then for every $5 - \delta < x < 5$, we have $\frac{x^2+1}{x-5} = -\frac{x^2+1}{5-x} < -\frac{x^2+1}{\delta} < -\frac{1}{\delta} = M$.

   Prove $\lim_{x\to 5-} \frac{x^2+1}{x-5} = -\infty$. Let $M < 0$ be arbitrary. Take $\delta = -\frac{1}{M}$. Then for every $5 - \delta < x < 5$, we have $\frac{x^2+1}{x-5} < \frac{1}{x-5} < \frac{1}{-\delta} = M$.

   Prove $\lim_{x\to 1} \sqrt{x+3} = 2$. Let $\varepsilon > 0$ be arbitrary. Take $\delta < \min\left\{\frac{\varepsilon}{2}, 1\right\}$. Write $h := x - 1$. Then we have, for $0 < |x-1| < \delta$, $0 < |h| < \delta$, and $\sqrt{x+3} = \sqrt{4+h} \leqslant \sqrt{4 + 8\,|h| + 4\,h^2} = 2\,(1 + |h|) = 2 + 2\,|h| < 2 + \varepsilon$; On the other hand $\sqrt{x+3} = \sqrt{4+h} \geqslant \sqrt{4-|h|} \geqslant \sqrt{4 - 8\,|h| + 4\,h^2} = 2 - 2\,|h| > 2 - \varepsilon$. Therefore for $0 < |x-1| < \delta$ we have $|\sqrt{x+3} - 2| < \varepsilon$.

3. Disprove $\lim_{n\to\infty} \frac{1}{n} = 2$. Take $\varepsilon = 1$. Let $N \in \mathbb{N}$ be arbitrary. Take $n = \max\{N, 1\}$. Then $n \geqslant N$ and $n \geqslant 1$. This leads to $\left|\frac{1}{n} - 2\right| \geqslant 1$ and the proof ends.

4. Disprove $\lim_{x\to 3} \frac{x^2+1}{x^2+x+5} = 2$. Take $\varepsilon = 1$. Let $\delta > 0$ be arbitrary. Take $x = 3 + r$ with $0 < |r| < \min\{\delta, 3\}$. Then $0 < |x-3| < \delta$ and $x > 0$. We have $\left|\frac{x^2+1}{x^2+x+5} - 2\right| = \left|\frac{x^2+2\,x+9}{x^2+x+5}\right| = \left|1 + \frac{x+4}{x^2+x+5}\right| = 1 + \frac{x+4}{x^2+x+5} \geqslant 1$ since $x > 0$. The proof ends.

5. Prove $\lim_{n\to\infty} \frac{\sin n^3}{n^3} = 0$. We have $-\frac{1}{n^3} \leqslant \frac{\sin n^3}{n^3} \leqslant \frac{1}{n^3}$. As $\lim_{n\to\infty}\left(-\frac{1}{n^3}\right) = \lim_{n\to\infty} \frac{1}{n^3} = 0$ the conclusion follows from Squeeze.

Exercise 5. Calculate the following limits. Justify your results.

$$\lim_{n\to\infty}\frac{(-4)^n+6^n}{5^{n+1}+6^{n+1}};\qquad \lim_{n\to\infty}\frac{n}{(2\,n^3+n)^{1/3}};\qquad \lim_{x\to+\infty}\frac{\sqrt{x+\sqrt{x}}}{\sqrt{x+1}}. \tag{5}$$

(Sol:[7];[8];[9])

Exercise 6. Let $a_n=(-1)^n-\frac{1}{n}$. Calculate $\sup_{n\in\mathbb{N}}a_n$, $\inf_{n\in\mathbb{N}}a_n$, $\limsup_{n\to\infty}a_n$, $\liminf_{n\to\infty}a_n$. Justify your answers. (Sol:[10])

- More exercises

Exercise 7. Let $a>0$, $b>1$. Prove the following.

$$\lim_{n\to\infty}\frac{n}{2^n}=0;\qquad \lim_{n\to\infty}\frac{1}{n^a}=0;\qquad \lim_{n\to\infty}a^{1/n}=1;\qquad \lim_{n\to\infty}\frac{n^a}{b^n}=0. \tag{6}$$

Exercise 8.

a) Let $a>0$ and $x_n=1+a+\frac{a^2}{2}+\frac{a^3}{3!}+\cdots+\frac{a^n}{n!}$. Prove $\{x_n\}$ is Cauchy. (Sol:[11])

b) Let $b\in\mathbb{R}$ and $x_n=1+b+b^2+\cdots+b^n$. Find all b such that $\{x_n\}$ is Cauchy. Justify.

c) Let $x_n=\frac{\sin 1}{1}+\cdots+\frac{\sin n^2}{n^2}$. Prove that $\{x_n\}$ is Cauchy.

Exercise 9. Given $\lim_{x\to 0+}\frac{\sin x}{x}=1$. Prove $\lim_{x\to 0-}\frac{\sin x}{x}=1$.

Exercise 10. Let $\{a_n\}$ be a sequence satisfying $\lim_{n\to\infty}a_n=a$ for some real number $a\neq 0$. Prove that there is $N\in\mathbb{N}$ such that for all $n\geqslant N$, $a_n\neq 0$.

Exercise 11. Let $\{a_n\},\{b_n\}$ be sequences with $\lim_{n\to\infty}a_n=-\infty, \lim_{n\to\infty}b_n=-\infty$. Prove or disprove:

$$\lim_{n\to\infty}(a_n\,b_n)=+\infty;\qquad \lim_{n\to\infty}(a_n+b_n)=-\infty;\qquad \lim_{n\to\infty}\frac{a_n}{b_n}=1. \tag{7}$$

(Sol: [12], [13])

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6. Prove $\lim_{n\to\infty}\frac{\sqrt{n^2-7}}{\sqrt{n^2+1}}=1$. Note that the sequence is only defined for $n\geqslant 3$. We have clearly $\frac{\sqrt{n^2-7}}{\sqrt{n^2+1}}<1$. On the other hand we have $\frac{\sqrt{n^2-7}}{\sqrt{n^2+1}}>\frac{\sqrt{n^2-7}}{\sqrt{(n+1)^2}}=\frac{\sqrt{(n-\sqrt{7})\,(n+\sqrt{7})}}{n+1}>\frac{n-\sqrt{7}}{n+1}$. The conclusion now follows from Squeeze.

7. $\lim_{n\to\infty}\frac{(-4)^n+6^n}{5^{n+1}+6^{n+1}}$. We have $\frac{(-4)^n+6^n}{5^{n+1}+6^{n+1}}=\frac{1}{6}\cdot\frac{\left(\frac{-4}{6}\right)^n+1}{\left(\frac{5}{6}\right)^{n+1}+1}$. The limit is $\frac{1}{6}$ after we prove that $\lim_{n\to\infty}r^n=0$ if $|r|<1$. Let $\varepsilon>0$ be arbitrary. Take $N>\log_r\varepsilon$. Then for every $n\geqslant N$, $r^n<r^N=\varepsilon$.

8. $\lim_{n\to\infty}\frac{n}{(2\,n^3+n)^{1/3}}$. We have $\frac{n}{(2\,n^3+n)^{1/3}}<\frac{n}{(2\,n^3)^{1/3}}=\frac{1}{2^{1/3}}$. On the other hand $\frac{n}{(2\,n^3+n)^{1/3}}>\frac{n}{[2\,(n+1)^3]^{1/3}}=\frac{1}{2^{1/3}}\,\frac{n}{n+1}$. Thus the limit is $2^{-1/3}$ thanks to Squeeze.

9. $\lim_{x\to+\infty}\frac{\sqrt{x+\sqrt{x}}}{\sqrt{x+1}}$. We prove that the limit is 1. We have $\frac{\sqrt{x+\sqrt{x}}}{\sqrt{x+1}}=\frac{\sqrt{1+x^{-1/2}}}{\sqrt{1+x^{-1}}}$. It suffices to prove $\lim_{x\to+\infty}\sqrt{1+x^{-1/2}}=\lim_{x\to+\infty}\sqrt{1+x^{-1}}=1$. As the proofs are almost identical I will only prove the first one here. Let $\varepsilon>0$ be arbitrary. Take $R>\varepsilon^{-2}$. Then for every $x>R$ we have $\sqrt{1+x^{-1/2}}<\sqrt{1+R^{-1/2}}<\sqrt{1+\varepsilon}<\sqrt{(1+\varepsilon)^2}<1+\varepsilon$. Consequently $\left|\sqrt{1+x^{-1/2}}-1\right|<\varepsilon$. We are done.

10. $\sup_{n\in\mathbb{N}}a_n=1$. To prove this, first we clearly have $(-1)^n-\frac{1}{n}<1$ so 1 is an upper bound. On the other hand for any $b<1$, taking $n=2\,k$ with $k>\frac{1}{1-b}$ we have $(-1)^n-\frac{1}{n}=1-\frac{1}{2\,k}>1-\frac{1-b}{2}=b+\frac{1-b}{2}>b$ so b is not an upper bound. Therefore $\sup_{n\in\mathbb{N}}a_n=1$.

$\limsup_{n\to\infty}a_n=1$. To prove this we estimate $\sup\{a_n,a_{n+1},\ldots\}$. Clearly $\forall n\in\mathbb{N}, a_n<1$ therefore $\sup\{a_n,\ldots\}\leqslant 1$. On the other hand, $\forall n\in\mathbb{N}$, $a_{2n}=1-\frac{1}{2\,n}\in\{a_n,a_{n+1},\ldots\}$ thus $\sup\{a_n,\ldots\}\geqslant 1-\frac{1}{2\,n}$. Now it follows from Squeeze that $\limsup_{n\to\infty}a_n=1$.

$\inf_{n\in\mathbb{N}}a_n=-2$. First $\forall n\in\mathbb{N}$, $(-1)^n-\frac{1}{n}\geqslant -1-1=-2$ so -2 is a lower bound. On the other hand, for any $b>-2$, we have $a_1=-1-1=-2<b$. Therefore -2 is the greatest lower bound.

$\liminf_{n\to\infty}a_n=-1$. The proof is similar to the limsup one.

11. Let $\varepsilon>0$ be arbitrary. Let $n_0>2\,a$. Denote $c:=\frac{a^{n_0}}{n_0!}$. Then we have for every $n>n_0$, $\frac{a^n}{n!}=\frac{a^{n-n_0}}{n\,(n-1)\cdots(n_0+1)}\,\frac{a^{n_0}}{n_0!}<c\left(\frac{1}{2}\right)^{n-n_0}$. Now take $N\in\mathbb{N}$ such that $c\left(\frac{1}{2}\right)^{N-n_0}<\varepsilon$. Let $m>n\geqslant N$ be arbitrary. We have $|x_m-x_n|=\left|\frac{a^{n+1}}{(n+1)!}+\cdots+\frac{a^m}{m!}\right|<c\left|\left(\frac{1}{2}\right)^{n+1-n_0}+\cdots+\left(\frac{1}{2}\right)^{m-n_0}\right|=c\left(\frac{1}{2}\right)^{n-n_0}\left[\frac{1}{2}+\cdots+\left(\frac{1}{2}\right)^{m-n}\right]<c\left(\frac{1}{2}\right)^{n-n_0}\times 2<c\left(\frac{1}{2}\right)^{N-n_0}<\varepsilon$.

12. Prove $\lim_{n\to\infty}(a_n\,b_n)=+\infty$. Let $M>0$ be arbitrary. As $\lim_{n\to\infty}a_n=-\infty$, there is $N_1>0$ such that $n\geqslant N_1\Longrightarrow a_n<-M^{1/2}$; Similarly there is $N_2\in\mathbb{N}$ such that $n\geqslant N_2\Longrightarrow b_n<-M^{1/2}$. Take $N=\max\{N_1,N_2\}$. Now for every $n\geqslant N$ we have $a_n\,b_n>(-M^{1/2})\cdot(-M^{1/2})=M$.

Exercise 12. Let $\{a_n\}$ be a sequence satisfying $\forall n\in\mathbb{N},\ a_n\in[0,1]$. Further assume $\lim_{n\to\infty}a_n=a$. Prove or disprove: $a\in[0,1]$. What if we replace $[0,1]$ by $(0,1)$ or $(0,1]$?

Exercise 13. Let $\{a_n\}$ be a sequence. Prove that $\lim_{n\to\infty}a_n$ exists if and only if for every subsequence $\{a_{n_k}\}$, $\lim_{k\to\infty}a_{n_k}$ exists. (Note that we do not assume the subsequence limits are the same).

Exercise 14. Let $\{a_n\},\{b_n\}$ be sequences such that $\forall n\in\mathbb{N}, b_n\neq 0$. Assume that $\lim_{n\to\infty}\frac{a_n}{b_n}=1$. Prove or disprove: $\lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n$.

Exercise 15. In Exercise 14, what if we further assume the existence of $\lim_{n\to\infty}a_n$ or $\lim_{n\to\infty}b_n$? That is, assume that $\lim_{n\to\infty}\frac{a_n}{b_n}=1$ and $\lim_{n\to\infty}a_n$ exists – could be either a number of $\pm\infty$ (or further assume $\lim_{n\to\infty}b_n$ exists), prove or disprove: $\lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n$.

Exercise 16. Prove or disprove: Let $\{a_n\},\{b_n\}$ be bounded sequences and further assume $\lim_{n\to\infty}a_n$ exists. Then: $\lim_{n\to\infty}b_n$ does not exist $\Longrightarrow \lim_{n\to\infty}(a_n\,b_n)$ does not exist.

Exercise 17. Prove or disprove: If $f(x)$ is not bounded, then there is $\{x_n\}$ such that $\lim_{n\to\infty}|f(x_n)|=+\infty$.

Exercise 18. Given that $\lim_{x\to 0}\sin x=0$ and $\lim_{x\to 0}\cos x=1$. Prove that for every $x_0\in\mathbb{R}$ there holds $\lim_{x\to x_0}\sin x=\sin x_0$.

Exercise 19. Given that $\lim_{x\to 0}\frac{\sin x}{x}=1$. Prove $\lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2}$. (Sol:[14])

Exercise 20. Let $\{a_n\}$ be a sequence. Prove or disprove: If $\lim_{n\to\infty}\frac{a_n+a_{n+1}+a_{n+2}}{3}=a$, then $\lim_{n\to\infty}a_n=a$.

Exercise 21. Let $\{a_n\}$, $\{b_n\}$ be sequences. Prove that $\limsup_{n\to\infty}(a_n+b_n)\leqslant\limsup_{n\to\infty}a_n+\limsup_{n\to\infty}b_n$. Can we replace $\leqslant$ by $=$?

Exercise 22. Let $\{a_n\}$ be a bounded sequence satisfying

$$\forall\{b_n\},\qquad \limsup_{n\to\infty}(a_n+b_n)=\limsup_{n\to\infty}a_n+\limsup_{n\to\infty}b_n, \tag{8}$$

then $\{a_n\}$ is convergent. (Hint:[15]) (Sol:[16])

- Problems.

Problem 1. Prove that $\lim_{n\to\infty}n^2\,2^{-n}=0$. (Hint:[17])

Problem 2. Let $n\in\mathbb{N}$ and $P(x)=x^n+a_{n-1}\,x^{n-1}+\cdots+a_1\,x+a_0$ be a polynomial of degree n. Prove that $\lim_{x\to+\infty}P(x)=+\infty$. (Hint:[18])

Problem 3. Let $\{a_n\}$ be a sequence satisfying $\lim_{n\to\infty}(a_{n+1}-a_n)=a\in\mathbb{R}$. Prove that $\lim_{n\to\infty}\frac{a_n}{n}=a$. (Hint:[19])

Problem 4. Let $E_n:=1+1+\frac{1}{2!}+\cdots+\frac{1}{n!}$. Prove that $\lim_{n\to\infty}E_n=e:=\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m$. (Hint:[20])

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13. Prove $\lim_{n\to\infty}(a_n+b_n)=-\infty$. Let $M<0$ be arbitrary. As $\lim_{n\to\infty}a_n=-\infty$ there is $N_1\in\mathbb{N}$ such that $\forall n\geqslant N_1$, $a_n<M$; As $\lim_{n\to\infty}b_n=-\infty$ there is $N_2\in\mathbb{N}$ such that $\forall n\geqslant N_2$, $b_n<M$. Now take $N=\max\{N_1,N_2\}$. For every $n\geqslant N$, we have $a_n+b_n<M+M=2\,M<M$.

14. We note that $1-\cos x=2\left(\sin\frac{x}{2}\right)^2$. Thus it suffices to prove $\lim_{x\to 0}\frac{\sin(x/2)}{x}=\frac{1}{2}$. Let $\varepsilon>0$ be arbitrary. As $\lim_{x\to 0}\frac{\sin x}{x}=1$ there is $\delta_0>0$ such that for every $0<|x|<\delta_0$, $\left|\frac{\sin x}{x}-1\right|<\varepsilon$. Take $\delta=2\,\delta_0$. Then for every $0<|x|<\delta$, we have $0<\left|\frac{x}{2}\right|<\delta_0$ and consequently $\left|\frac{\sin(x/2)}{x/2}-1\right|<\varepsilon\Longrightarrow\left|\frac{\sin(x/2)}{x}-\frac{1}{2}\right|<\frac{\varepsilon}{2}<\varepsilon$.

15. Take $b_n=-a_n$.

16. Take $b_n=-a_n$ we have $0=\limsup_{n\to\infty}0=\limsup_{n\to\infty}(a_n+b_n)=\limsup_{n\to\infty}a_n+\limsup_{n\to\infty}(-a_n)=\limsup_{n\to\infty}a_n-\liminf_{n\to\infty}a_n$. Thus $\limsup_{n\to\infty}a_n=\liminf_{n\to\infty}a_n$ and convergence follows.

17. Expand 2^n using binomial expansion.

18. Let $M>0$ be arbitrary. Take $R>1$ such that $R>|a_0|+|a_1|+\cdots+|a_{n-1}|+M$. Then $x>R$ implies $P(x)>M$.

19. Write $\frac{a_n}{n}=\frac{(a_n-a_{n-1})+(a_{n-1}-a_{n-2})+\cdots+(a_2-a_1)+a_1}{n}$. Let $\varepsilon>0$ be arbitrary. Let $N_0\in\mathbb{N}$ be such that $|a_n-a_{n-1}-a|<\frac{\varepsilon}{2}$ whenever $n\geqslant N_0$. Now write $\frac{a_n}{n}=\frac{(a_n-a_{n-1})+\cdots+(a_{N_0+1}-a_{N_0})}{n}+\frac{(a_{N_0}-a_{N_0-1})+\cdots+(a_2-a_1)+a_0}{n}$. Find appropriate N.

20. Recall that we have proved (in the lecture on Sept. 24) $E_n>\left(1+\frac{1}{n}\right)^n$. On the other hand, try to prove $E_n\leqslant\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m$.

Problem 5. Let $p>0$ and $H_n:=1+\frac{1}{2^p}+\frac{1}{3^p}+\cdots+\frac{1}{n^p}$. Prove that $\{H_n\}$ converges if and only if $p>1$. (Hint:[21])

21. When $p\leqslant 1$ the divergence is clear as $1+\frac{1}{2^p}+\cdots+\frac{1}{n^p}>1+\frac{1}{2}+\cdots+\frac{1}{n}$. When $p>1$ form groups of sizes $1,2,4,8,\ldots..$