Math 117 Fall 2014 Midterm 1 Review

- Midterm 1 coverage:
  - Lectures 1 - 11 and the exercises therein.
  - Required sections in Dr. Bowman's book and my 314 notes.
  - Homeworks 1 & 2.
  - The exercises below are to help you on the concepts and techniques. The exam problems may or may not look like them.
  - Pages 3, 4, 6 of the Midterm Review of Math 314, 2013 may also help.
- Important topics and requirements:
  - Numbers.
    - Prove a certain number is rational/irrational.

Exercise 1. Prove that $\sqrt{13}$ is irrational.

Exercise 2. Prove that $\sqrt{15}+\sqrt{7}$ is irrational.

Exercise 3. Prove that $\sqrt{2}+\sqrt[3]{4}$ is irrational. (Sol:[1])

Exercise 4. Prove that

$$\sqrt{5},\ \sqrt{5\sqrt{5}},\ \sqrt{5\sqrt{5\sqrt{5}}},\ \ldots. \tag{1}$$

are all irrational.

Exercise 5. Can you find two irrational numbers a, b such that $a\,b$ and a/b are both rational?

Problem 1. Let $n \in \mathbb{N}$. Prove that $\sqrt{n\sqrt{n+1}} \notin \mathbb{Q}$. (Hint:[2])

    - Upper/lower bounds; Sup and Inf.

Exercise 6. Let $A \subseteq \mathbb{R}$ and m be a lower bound of A. Prove that any $m' < m$ is also a lower bound of A. (Sol:[3])

Exercise 7. Let $A = \left\{1 - \frac{1}{n} | n \in \mathbb{N}\right\}$. Find $\sup A, \inf A$ and justify.

Exercise 8. Let $A \subseteq \mathbb{R}$. If there is $a \in A$ such that $a \geqslant a'$ for every $a' \in A$, we say a is the maximum of the set A and write $\max A = a$. Prove that if $\max A$ exists, then $\sup A = \max A$. (Sol:[4])

Exercise 9. Prove that for $A = \left\{1 - \frac{1}{n} | n \in \mathbb{N}\right\}$, $\max A$ does not exist.

Problem 2. Find out the value of

$$\sqrt{5\sqrt{5\sqrt{5\sqrt{\cdots}}}} \tag{2}$$

through proving certain sequence of numbers is increasing and has an upper bound. (Hint:[5])

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1. Assume $a = \sqrt{2} + \sqrt[3]{4} \in \mathbb{Q}$. Then $4 = (a - \sqrt{2})^3 = a^3 - 3\,a^2\,\sqrt{2} + 6\,a - 2\,\sqrt{2}$ which gives $\sqrt{2} = \frac{a^3 + 6\,a - 4}{3\,a^2 + 2} \in \mathbb{Q}$. Contradiction.

2. As this is a "Problem" which is supposed to be hard, I won't give full solution, but just a few key steps. Assume $\sqrt{n\sqrt{n+1}} \in \mathbb{Q}$. Then $\sqrt{n+1} \in \mathbb{Q}$. This means $n+1 = m^2$ for some $m \in \mathbb{N}$. Consequently $\sqrt{n\,m} \in \mathbb{Q}$. Now notice that $(n, m) = 1$. Therefore necessary $\sqrt{n} \in \mathbb{Q}$ and $\sqrt{m} \in \mathbb{Q}$. But $\sqrt{n} \in \mathbb{Q}$ implies $n = k^2$ for some $k \in \mathbb{N}$. But it is possible to have both $n, n+1$ squares as $|m^2 - k^2| = |m+k|\,|m-k| \geqslant |1+1| \cdot 1 = 2$.

3. Take an arbitrary $a \in A$, we will prove $m' \leqslant a$ and then by definition m' is a lower bound of A. As m is a lower bound of A, we have $m \leqslant a$. But then $m' < m \leqslant a$ which means $m' \leqslant a$ and thus ends the proof.

4. To prove $\sup A = \max A = a$, recalling the definition of sup, we see that we need to prove two things: For every $a' \in A$, $a \geqslant a'$. For any $b < a$, there is $a' \in A$ such that $b < a'$. We first prove the first. Take an arbitrary $a' \in A$, as $a = \max A$ we have $a \geqslant a'$. Now take an arbitrary $b < a$. Take $a' = a \in A$. Then we have $b < a'$. Therefore $\sup A = a$.

Problem 3. Let $A \subseteq \mathbb{R}$ and define $B := \{-x | x \in A\}$. Prove that $\sup B = -\inf A$. (Hint:[6])

- Sets.
  - Prove relations between abstract sets;

Exercise 10. Let A, B, C be sets. Prove

$$(C - A) \cap (C - B) = C - (A \cup B) \tag{3}$$

and

$$(C - A) \cup (C - B) = C - (A \cap B). \tag{4}$$

(Sol:[7])

Exercise 11. Let A, B, C be sets. Is it always true that

$$(A \cap B) \cup C = A \cap (B \cup C)? \tag{5}$$

Justify your answer.

  - Intervals.

Exercise 12. Let $a, b, c, d \in \mathbb{R}$ with $a < b$, $c < d$. Prove that $(a, b) \subseteq [c, d]$ if and only if $(a, b) \subseteq (c, d)$. Is it true that $(a, b) \subset [c, d]$ if and only if $(a, b) \subset (c, d)$? Justify your answer.

Exercise 13. Let $A = [0, 1]$ and $B = (1, 2)$. Calculate $A \cap B$ and $A \cup B$. Justify your answers.

Exercise 14. Calculate $\cap_{n \in \mathbb{N}} (n, +\infty)$ and $\cap_{n \in \mathbb{N}} [n, +\infty)$. Justify your answers. (Sol:[8])

Exercise 15. Calculate $\cup_{n \in \mathbb{N}} (-n, n)$ and $\cup_{n \in \mathbb{N}} [-n, n]$. Justify your answers. (Sol:[9])

- Functions.
  - Prove the relations in §3.2 of the note "Sets and Functions".
  - Composite and inverse functions.

Exercise 16. Let $f_1(x) = x^2$, $f_2(x) = x^3$, $f_3(x) = x^{-1}$. Calculate $(f_1 \circ f_2 \circ f_3)(2)$, $(f_2 \circ f_1 \circ f_3)(2)$, $(f_3 \circ f_2 \circ f_1)(2)$.

Exercise 17. A function f is called "increasing" if whenever $x < y$ there holds $f(x) \leqslant f(y)$. A function f is called "strictly increasing" if whenever $x < y$ there holds $f(x) < f(y)$.

a) Define "decreasing" and "strictly decreasing" functions.

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5. Set $x_n := \sqrt{5\sqrt{5\sqrt{\cdots\sqrt{5}}}}$ where there are n square roots. Then we have $x_n = 5^{(1/2 + 1/4 + \cdots + 1/2^n)} = 5^{1 - 2^{-n}}$ and the conclusions follow. Alternatively, we have $x_{n+1} = \sqrt{5\sqrt{5 \cdots \sqrt{5\sqrt{5}}}} > \sqrt{5\sqrt{5\sqrt{\cdots\sqrt{5}}}}$ (the last $\sqrt{5}$ replaced by 1. And then we use induction to prove $x_n < 5$ for all n.

6. We prove 1. for every $b \in B$, $b \leqslant -\inf A$; 2. for any $m < -\inf A$ there is $b \in B$ such that $b > m$.

For the first claim, take an arbitrary $b \in B$. By definition of B there is $a \in A$ such that $b = -a$. Now we have $a \geqslant \inf A$ which gives $b = -a \leqslant -\inf A$.

For the second claim, take an arbitrary $m < -\inf A$. Then we have $-m > \inf A$. Thus there is $a \in A$ such that $a < -m$. Taking $b = -a \in B$ we have $b = -a > -(-m) = m$.

7. We prove the first one. Recall that to prove "=" we need to prove "$\subseteq$" and "$\supseteq$".

First we prove $(C - A) \cap (C - B) \subseteq C - (A \cup B)$. Take an arbitrary $x \in (C - A) \cap (C - B)$. Then $x \in C - A$ and $x \in C - B$. This gives $x \in C, x \notin A, x \in C, x \notin B$ which means $x \in C, x \notin A \cup B$ and consequently $x \in C - (A \cup B)$.

Next we prove $C - (A \cup B) \subseteq (C - A) \cap (C - B)$. Take an arbitrary $x \in C - (A \cup B)$. Then $x \in C$, $x \notin A \cup B$. But if $x \notin A \cup B$ then $x \notin A$ which means $x \in C - A$. A similar argument gives $x \in C - B$. Therefore $x \in (C - A) \cap (C - B)$.

8. First guess $\cap_{n \in \mathbb{N}} (n, +\infty) = \varnothing$. Next we prove this claim. Take any $x \in \mathbb{R}$. There is $n_0 \in \mathbb{N}$ such that $n_0 > x$. Then by definition we have $x \notin (n_0, +\infty)$. By definition of $\cap_{n \in \mathbb{N}} (n, +\infty)$ we see that $x \notin \cap_{n \in \mathbb{N}} (n, +\infty)$. Thus there is no number in this set and it must be $\varnothing$.

The proof of $\cap_{n \in \mathbb{N}} [n, +\infty)$ is almost identical.

9. We guess $\cup_{n \in \mathbb{N}} (-n, n) = \mathbb{R}$. To prove, take any $x \in \mathbb{R}$. There is $n_0 \in \mathbb{N}$ such that $n_0 > |x|$. Then $x \in (-n_0, n_0)$ and therefore $x \in \cup_{n \in \mathbb{N}} (-n, n)$. Consequently $\mathbb{R} \subseteq \cup_{n \in \mathbb{N}} (-n, n)$. On the other hand, take an arbitrary $x \in \cup_{n \in \mathbb{N}} (-n, n)$ then by definition of $\cup$ there is $n_0 \in \mathbb{N}$ such that $x \in (-n_0, n_0)$ which through definition of intervals implies $x \in \mathbb{R}$. Therefore $x \in \mathbb{R}$ and we have $\cup_{n \in \mathbb{N}} (-n, n) \subseteq \mathbb{R}$. Summarizing, we have proved $\cup_{n \in \mathbb{N}} (-n, n) = \mathbb{R}$.

b) Find one example for each of the four types of functions.

c) Prove: If a function is strictly increasing or strictly decreasing, then it is one-to-one. Does the conclusion still hold if we discard "strictly"?

Exercise 18. Let $x, y \in \mathbb{R}$. Apply triangle inequality to prove

$$|x| - |y| \leqslant |x - y|. \tag{6}$$

(Sol:[10])

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10. By triangle inequality we have

$$|x| = |(x - y) + y| \leqslant |x - y| + |y|. \tag{7}$$

the conclusion immediately follows.