MATH 117 FALL 2014 FINAL REVIEW PROBLEMS

- Final exam coverage:
  - Lectures 1 – 48 and the exercises therein.
  - Required sections in Dr. Bowman's book and my 314 notes.
  - Homeworks 1 – 9.
  - The exercises below only cover materials after Midterm 3. You should also study the Review Problems for Midterms 1 – 3.
  - The exercises below are to help you on the concepts and techniques. The exam problems may or may not look like them.
- Exercises.

Exercise 1. What is the difference between $\int_a^b f(x)\,\mathrm{d}x$ and $\int_a^b f(t)\,\mathrm{d}t$? (Sol:[1])

Exercise 2. Prove **by definition** the integrability of $f := \begin{cases} 1 & x > 0 \\ -1 & x \leqslant 0 \end{cases}$ on $[-1, 1]$ and find $\int_{-1}^1 f(x)\,\mathrm{d}x$.

Exercise 3. Calculate

$$\int_0^1 x^7\,\mathrm{d}x; \quad \int_0^1 \frac{\mathrm{d}x}{1+x^2}; \quad \int_0^{\pi/2} \cos x\,\mathrm{d}x; \quad \int_0^{\pi/4} \frac{\mathrm{d}x}{(\cos x)^2}. \tag{1}$$

Justify your calculation.

Exercise 4. Calculate

$$\left(\int_0^x \sqrt{\frac{1-t}{1+t^2}}\,\mathrm{d}t\right)'; \quad \left(\int_{\cos x}^{x^7} e^{-\sin t}\,\mathrm{d}t\right)'. \tag{2}$$

Justify your calculation. (Sol:[2])

Exercise 5. Let $f(x) := e^{\sin x}$. Calculate $f', f'', f''', f^{(4)}$ and then the Taylor expansion with Lagrange form of remainder at $x_0 = 0$ to degree 3 (that is $n = 3$). (Sol:[3])

Exercise 6. Calculate

$$\lim_{x\to 0} \frac{\sin 5x}{\sin 3x}; \qquad \lim_{x\to 0} \frac{\cos x - \cos 100x}{x^2}; \qquad \lim_{x\to 0} \frac{x(1-\cos x)}{\sin(x^3)}. \tag{4}$$

(Hint:[4])

Exercise 7. Find all $x \in \mathbb{R}$ such that $\sum_{n=1}^\infty \frac{x^n}{n^3}$ is convergent. Justify. (Sol:[5])

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1. Nothing. They are always equal.

2. For the first one we apply FTC2 to obtain the derivative to be $\sqrt{(1-x)/(1+x^2)}$ (should add assumption: $x < 1$, sorry!) For justification just notice that for all $t \in [0, x]$ the function $\sqrt{(1-t)/(1+t^2)}$ is continuous. For the second one define $G(x) := \int_0^x e^{-\sin t}\,\mathrm{d}t$. Then we are calculating $[G(x^7) - G(\cos x)]' = G'(x^7)\,(x^7)' - G'(\cos x)\,(\cos x)' = e^{-\sin x^7}\,7\,x^6 - e^{-\sin(\cos x)}\,(-\sin x)$. Justification is trivial as $e^{-\sin t}$ is continuous on $\mathbb{R}$.

3. We have $f' = e^{\sin x}\cos x$, $f'' = e^{\sin x}\cos^2 x - e^{\sin x}\sin x$, $f''' = e^{\sin x}\cos^3 x - 3\,e^{\sin x}\sin x\cos x - e^{\sin x}\cos x$, $f^{(4)} = 3\,e^{\sin x}\sin^2 x + e^{\sin x}\sin x + e^{\sin x}\cos^4 x - 4\,e^{\sin x}\cos^2 x - 6\,e^{\sin x}\sin x\cos^2 x$. Thus $f(0) = 1$, $f'(0) = 1$, $f''(0) = 1$, $f'''(0) = 0$, and

$$e^{\sin x} = 1 + x + \frac{x^2}{2} + \frac{x^4}{24}\,e^{\sin c}\,[3\sin^2 c + \sin c + \cos^4 c - 4\cos^2 c - 6\sin c\cos^2 c] \tag{3}$$

where $c \in (0, x)$.

4. For $\lim_{x\to 0} \frac{x(1-\cos x)}{\sin(x^3)}$, apply L'Hospital 3 times.

5. First we calculate the radius of convergence:

$$\rho := \left(\limsup_{n\to\infty} \left(\frac{1}{n^3}\right)^{1/n}\right)^{-1} = 1. \tag{5}$$

Therefore the power series converges when $|x| < 1$ and diverges when $|x| > 1$. For $|x| = 1$, we check $\left|\frac{x^n}{n^3}\right| \leqslant \frac{1}{n^3}$. As $\sum_{n=1}^\infty \frac{1}{n^3}$ converges, so does $\sum_{n=1}^\infty \frac{x^n}{n^3}$. Summarizing, the series converges for $|x| \leqslant 1$ and diverges for $|x| > 1$.

- More exercises

Exercise 8. Let $f: [0,1] \mapsto \mathbb{R}$ be integrable and $c \in \mathbb{R}$. Prove **by definition** that $c\, f$ is integrable on $[0,1]$ and furthermore

$$\int_0^1 (c\, f)(x)\, \mathrm{d}x = c \int_0^1 f(x)\, \mathrm{d}x. \tag{6}$$

Exercise 9. Let $m \in \mathbb{N}$. Let $f: [0,n] \mapsto \mathbb{R}$ be integrable. Let $g(x) := f(m\, x)$. Prove **by definitio**n that g is integrable on $[0,1]$ and calculate $\int_0^1 g(x)\, \mathrm{d}x$. (Sol:[6])

Exercise 10. Let $f: [0,1] \mapsto \mathbb{R}$ be continuous and non-negative. Prove: If $\int_0^1 f(x)\, \mathrm{d}x = 0$ then $f(x) = 0$ for all $x \in [0,1]$. Does the conclusion still hold if we drop the assumption of continuity? (Sol:[7])

Exercise 11. Find a function $f: [0,1] \mapsto \mathbb{R}$ such that $|f|$ is integrable on $[0,1]$ but f is not. Justify.

Exercise 12. Let $f(x) := \begin{cases} x^4 \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$. Find the largest $k \in \mathbb{N}$ such that $f(x)$ is k-th differentiable for all $x \in \mathbb{R}$. Justify. (Sol:[8])

Exercise 13. Let $f(x) = x^2 \cos x$. Calculate $f^{(50)}(x)$. (Ans:[9])

Exercise 14. Does $\lim_{x \to 0+} x^x$ exist? If it does find the value. Here $x^x := e^{x \ln x}$. (Sol:[10])

Exercise 15. Prove: If $f''(0)$ exists, then

$$\lim_{h \to 0} \frac{f(h) + f(-h) - 2\, f(0)}{h^2} = f''(0). \tag{10}$$

(Sol:[11])

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6. First it should be $f: [0,m] \mapsto \mathbb{R}$. We prove $U(g) = \frac{1}{m} U(f)$. Similarly $L(g) = \frac{1}{m} L(f)$. As f is integrable we have $U(f) = L(f)$ and consequently $U(g) = L(g) = \frac{1}{m} \int_0^m f(x)\, \mathrm{d}x$ which yields the integrability of g as well as $\int_0^1 g(x)\, \mathrm{d}x = \frac{1}{m} \int_0^m f(x)\, \mathrm{d}x$.

We first prove $U(g) \leqslant \frac{1}{m} U(f)$. Let $P = \{x_0, ..., x_n\}$ be an arbitrary partition of $[0,m]$. Then $P' = \{\frac{x_0}{m}, ..., \frac{x_n}{m}\}$ is a partition of $[0,1]$. We calculate

$$U(g, P') = \sum_{k=1}^n \sup_{[x_{k-1}/m, x_k/m]} g(x) \left(\frac{x_k}{m} - \frac{x_{k-1}}{m}\right) = \frac{1}{m} \sum_{k=1}^n \sup_{[x_{k-1}/m, x_k/m]} f(m\, x)\, (x_k - x_{k-1}) = \frac{1}{m} U(f, P). \tag{7}$$

This gives $U(g) \leqslant \frac{1}{m} U(f, P)$. As P is arbitrary, we have $U(g) \leqslant \frac{1}{m} \inf_P U(f, P) = \frac{1}{m} U(f)$.

Next we prove $U(g) \geqslant \frac{1}{m} U(f)$. Let $P = \{x_0, ..., x_n\}$ be an arbitrary partition of $[0,1]$. Then $P' = \{m\, x_0, ..., m\, x_n\}$ is a partition of $[0,m]$. Thus we have

$$U(g, P) = \sum_{k=1}^n \sup_{[x_{k-1}, x_k]} g(x)\, (x_k - x_{k-1}) = \sum_{k=1}^n \sup_{[m x_{k-1}, m x_k]} f(x)\, (x_k - x_{k-1}) = \frac{1}{m} U(f, P') \geqslant \frac{1}{m} U(f). \tag{8}$$

Taking infimum over all P we have $U(g) \geqslant \frac{1}{m} U(f)$.

7. Assume the contrary. Then there is $c \in [0,1]$ such that $f(c) > 0$. As f is continuous, there is $\delta > 0$ such that $f(x) > f(c)/2$ for all $x \in [c - \delta, c + \delta] \cap [0,1]$. Denote $[a,b] := [c - \delta, c + \delta] \cap [0,1]$. Then $b - a \geqslant \delta$. We have

$$\int_0^1 f(x)\, \mathrm{d}x \geqslant \int_a^b f(x)\, \mathrm{d}x \geqslant \int_a^b \frac{f(c)}{2}\, \mathrm{d}x \geqslant \frac{f(c)\, \delta}{2} > 0 \tag{9}$$

contradiction. The conclusion is false if f is not assumed to be continuous. For example $f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$ is not zero for every $x \in [0,1]$ but $\int_0^1 f(x)\, \mathrm{d}x = 0$.

8. First study $k = 1$. Clearly $f(x)$ is differentiable at every $x \neq 0$ with $f'(x) = 4\, x^3 \sin\frac{1}{x} - x^2 \cos\frac{1}{x}$. On the other hand we have $\lim_{x \to 0} \frac{x^4 \sin(1/x)}{x} = 0$ so $f'(0) = 0$.

Next study $k = 2$. Clearly $f'(x)$ is differentiable at every $x \neq 0$ with $f''(x) = 12\, x^2 \sin\frac{1}{x} - 6\, x \cos\frac{1}{x} - \sin\frac{1}{x}$. On the other hand we have $\lim_{x \to 0} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \to 0} \left[4\, x^2 \sin\frac{1}{x} - x \cos\frac{1}{x}\right] = 0$ so $f''(0) = 0$.

Now for $k = 3$. We notice $\lim_{x \to 0} f''(x)$ does not exist thus $f''(x)$ is not continuous at 0 and consequently is not differentiable at 0. So $f'''(0)$ does not exist.

So the largest k is 2.

9. $(2450 - x^2) \cos x - 100\, x \sin x$. x

10. We have $\lim_{x \to 0+} x \ln x = \lim_{t \to \infty} t\, e^{-t} = 0$ by L'Hospital. Therefore $\lim_{x \to 0+} x^x = \lim_{x \to 0+} e^{x \ln x} = e^{\lim_{x \to 0+} x \ln x} = e^0 = 1$. Note that we have use the continuity of e^x.

- Problems.

Problem 1. Let $f: [a, b] \mapsto [c, d]$ and $g: [c, d] \mapsto \mathbb{R}$. Prove or disprove:

a) If f is integrable on $[a, b]$ and g is continuous on $[c, d]$, then $g \circ f$ is integrable on $[a, b]$;

b) If f is integrable on $[a, b]$ and g is integrable on $[c, d]$, then $g \circ f$ is integrable on $[a, b]$.

Problem 2. Let $f: [0, 1] \mapsto \mathbb{R}$ be Riemann integrable. Prove or disprove

$$\int_a^b f(x) \, \mathrm{d}x = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right). \tag{12}$$

(Hint:[12])

Problem 3. Prove or disprove:

i. If $|f(x) - c|$ is Riemann integrable on $[a, b]$ for all $c \in \mathbb{R}$, then $f(x)$ is Riemann integrable on $[a, b]$;

ii. If $|f(x)|$ and $|f(x) - 1|$ are Riemann integrable on $[a, b]$, then f is Riemann integrable on $[a, b]$.

Problem 4. Prove that, $\forall x \in \mathbb{R}$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{15}$$

using the following idea by Jyesthadeva (c. 1500 - c. 1575) of ancient India:

i. It suffices to prove for $x > 0$;

ii. When $x > 0$ there holds $0 < \sin x < x$;

iii. Now apply

$$\cos x = \int_0^x \sin t \, \mathrm{d}t; \qquad \sin x = 1 - \int_0^x \cos t \, \mathrm{d}t \tag{16}$$

again and again. (Hint:[13])

Problem 5. Let $h(x): [0, 1] \mapsto \mathbb{R}$ be such that $h(x) > 0$ for every $x \in [0, 1]$. Prove that there is a partition $\{x_0 = 0, x_1, \ldots, x_n = 1\}$ and $t_k \in [x_{k-1}, x_k]$, $k = 1, 2, \ldots, n$ such that

$$(x_k - x_{k-1}) < h(t_k). \tag{19}$$

(Hint:[14])

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11. By Taylor's theorem we have $f(x) = f(0) + f'(0)\, x + \frac{f''(0)}{2} x^2 + R(x)$ with $\lim_{x \to 0} \frac{R(x)}{x^2} = 0$. Substituting this into the formula we have

$$\frac{f(h) + f(-h) - 2\, f(0)}{h^2} = f''(0) + \frac{R(h) + R(-h)}{h^2} \longrightarrow f''(0) \quad \text{as } h \longrightarrow 0. \tag{11}$$

12. Let $\varepsilon > 0$ be arbitrary. As f is integrable there is a partition $P = \{x_0, \ldots, x_m\}$ of $[0, 1]$ such that $U(f, P) - \int_0^1 f < \varepsilon/2$, $\int_0^1 f - L(f, P) < \varepsilon/2$. Now for each n, denote $P_n = \left\{0, \frac{1}{n}, \ldots, 1\right\}$. We have

$$U(f, P_n) = \sum_{k: \left[\frac{k-1}{n}, \frac{k}{n}\right] \subseteq [x_{l-1}, x_l] \text{ for some } l} \left(\sup_{\left[\frac{k-1}{n}, \frac{k}{n}\right]} f\right) \frac{1}{n} + \sum_{\text{other } k\text{'s}} \left(\sup_{\left[\frac{k-1}{n}, \frac{k}{n}\right]} f\right) \frac{1}{n}. \tag{13}$$

Now note that there are at most $2\, m$ terms in the 2nd sum so it is no more than $\frac{2\, m\, M}{n}$ where $M := \sup_{[0,1]} |f|$ which is finite as f, being integrable, must be bounded. Now prove

$$U(f, P_n) \leqslant U(f, P) + \frac{4\, m\, M}{n} \tag{14}$$

and similarly $L(f, P_n) \geqslant L(f, P) - \frac{4\, m\, M}{n}$.

13. Obviously (16) is nonsense! It should be

$$\sin x = \int_0^x \cos t \, \mathrm{d}t; \qquad \cos x = 1 - \int_0^x \sin t \, \mathrm{d}t. \tag{17}$$

Now as $0 < \sin x < x$, we have $0 < \int_0^x \sin t \, \mathrm{d}t < \int_0^x t \, \mathrm{d}t = \frac{x^2}{2}$. Thus

$$1 > \cos x > 1 - \frac{x^2}{2}. \tag{18}$$

But then $\int_0^x 1 \, \mathrm{d}t > \int_0^x \cos t \, \mathrm{d}t > \int_0^x \left(1 - \frac{t^2}{2}\right) \mathrm{d}t = x - \frac{x^3}{6}$. So $\sin x > x - \frac{x^3}{6}$. Substitute into $\cos x = 1 - \int_0^x \sin t \, \mathrm{d}t$ we have $\cos x < 1 - \frac{x^2}{2} + \frac{x^4}{24}$. And so on. To prove convergence, prove that the series is Cauchy.

Note. This leads to a generalization of Riemann integration by Jaroslav Kurzweil and Ralph Henstock around 1960. See "Return to the Riemann Integral" by R. G. Bartle, American Mathematical Monthly, 1996.

Problem 6. Prove or disprove: Let f, g satisfy all assumptions of L'Hospital but instead of the existence of $\lim_{x \to x_0} \frac{f'(x)}{g'(x)}$ we have that $\lim_{x \to x_0} \frac{f'(x)}{g'(x)}$ does not exist, then $\lim_{x \to x_0} \frac{f(x)}{g(x)}$ does not exist.

Problem 7. Let $f: [0,1] \mapsto \mathbb{R}$ be integrable and $g: \mathbb{R} \mapsto \mathbb{R}$ be periodic with period 1[15] and integrable on $[0,1]$. Prove

$$\lim_{n \to \infty} \int_0^1 f(x)\, g(n\, x)\, \mathrm{d}x = \left(\int_0^1 f(x)\, \mathrm{d}x \right) \left(\int_0^1 g(x)\, \mathrm{d}x \right). \tag{20}$$

(Hint:[16])

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14. Prove by contradiction and use nested intervals. Note that all we need is $(x_k - x_{k-1}) < \sup_{[x_{k-1}, x_k]} h(x)$. Start from an arbitrary partition and then refine to try to meet this requirement. Refine in such a way that the size of the intervals tends to zero. Note that if no matter how we refine there is always an interval violating the requirement, we have a sequence of nested interval $[a_n, b_n]$ on which $b_n - a_n \geqslant \sup_{[a_n, b_n]} h(x)$. Let $\{c\} = \cap_{n=1}^{\infty} [a_n, b_n]$. We have $b_n - a_n \geqslant h(c)$ for all n. Taking $n \to \infty$ we have $0 \geqslant h(c)$. But we are given $h(c) > 0$. Contradiction.

15. That is $g(x+1) = g(x)$ for all $x \in \mathbb{R}$.

16. As f, g are both integrable there is M such that $|f|, |g| < M$ for all relevant x. Let $\varepsilon > 0$ be arbitrary. There is $m \in \mathbb{N}$ and $f_m(x) := \sup_{[(k-1)/m, k/m]} f$ for $x \in \left[\frac{k-1}{m}, \frac{k}{m}\right]$ such that $\int_0^1 |f(x) - f_m(x)|\, \mathrm{d}x < \frac{\varepsilon}{2\,M}$. Now find $N \in \mathbb{N}$ such that for all $n \geqslant N$,

$$\left| \int_0^1 f_m(x)\, g(n\, x)\, \mathrm{d}x - \int_0^1 f(x)\, \mathrm{d}x \int_0^1 g(x)\, \mathrm{d}x \right| < \frac{\varepsilon}{2\,M}. \tag{21}$$