Math 118 Winter 2015 Lecture 48 (Apr. 10, 2015)

Final Review III: Curves and Surfaces (Cont.)

- Solids of revolution.
  - $y = f(x) \geqslant 0, a \leqslant x \leqslant b$ rotate around x-axis.
$$V = \pi \int_a^b f(x)^2 \, \mathrm{d}x. \tag{1}$$
  - $y = f(x) \geqslant 0,\ a \leqslant x \leqslant b$ rotate around y-axis.
$$V = 2\pi \int_a^b x\, f(x) \, \mathrm{d}x. \tag{2}$$
  - Trivial generalizations to the case where we have two graphs $y = f(x), y = g(x)$.

Example 1. Calculate the volume of a solid generated by the revolution of the catenary $y = \frac{1}{2}(e^x + e^{-x}),\ 0 \leqslant x \leqslant 1$

a) around the x-axis;

b) around the y-axis.

Solution.

a) We have
$$\begin{aligned} V &= \pi \int_0^1 \left[\frac{e^x + e^{-x}}{2}\right]^2 \mathrm{d}x \\ &= \frac{\pi}{4} \int_0^1 [e^{2x} + e^{-2x} + 2] \, \mathrm{d}x \\ &= \frac{\pi}{8}(e^2 - e^{-2} + 4). \end{aligned} \tag{3}$$

b) We have
$$\begin{aligned} V &= 2\pi \int_0^1 x \left(\frac{e^x + e^{-x}}{2}\right) \mathrm{d}x \\ &= \pi \int_0^1 x\,(e^x + e^{-x}) \, \mathrm{d}x \\ &= \pi \int_0^1 x \, \mathrm{d}(e^x - e^{-x}) \\ &= \pi \left[x\,(e^x - e^{-x})|_0^1 - \int_0^1 (e^x - e^{-x}) \, \mathrm{d}x\right] \\ &= \pi\,[e - e^{-1} - (e + e^{-1} - 2)] \\ &= 2\pi\,(1 - e^{-1}). \end{aligned} \tag{4}$$

- Surfaces of revolution.
  - $y = f(x),\ a \leqslant x \leqslant b$ rotate around x-axis.
$$A = 2\pi \int_a^b f(x) \sqrt{1 + f'(x)^2} \, \mathrm{d}x = \int [\text{circumference}] \cdot [\text{infinitesimal arc length}]. \tag{5}$$

Example 2. Surface area of the surface of revolution generated by $y^2 = 2\,x$, $0 \leqslant x \leqslant 4$,

a) around x-axis;

b) around y-axis.

Solution.

a) We have $f(x) = \sqrt{2\,x}$ and $f'(x) = \frac{1}{\sqrt{2\,x}}$. Thus

$$\begin{aligned} A &= 2\,\pi \int_0^4 \sqrt{2\,x}\,\sqrt{1+\frac{1}{2\,x}}\,\mathrm{d}x \\ &= 2\,\pi \int_0^4 \sqrt{1+2\,x}\,\mathrm{d}x \\ &\overset{u=\sqrt{1+2x}}{=\!=\!=} 2\,\pi \int_1^3 u^2\,\mathrm{d}u \\ &= \frac{52\,\pi}{3}. \end{aligned} \tag{6}$$

b) Note that this is the same as rotating $y = \frac{x^2}{2}$ around x-axis for $-2\sqrt{2} \leqslant x \leqslant 2\sqrt{2}$. Therefore

$$\begin{aligned} A &= 2\,\pi \int_{-2\sqrt{2}}^{2\sqrt{2}} \frac{x^2}{2}\sqrt{1+x^2}\,\mathrm{d}x \\ &= 2\,\pi \int_0^{2\sqrt{2}} x^2\sqrt{1+x^2}\,\mathrm{d}x \\ &\overset{u=x^2}{=\!=\!=} \pi \int_0^8 u^{1/2}\,(1+u)^{1/2}\,\mathrm{d}u. \end{aligned} \tag{7}$$

Recalling the Chebyshev theorem of indefinite integral, we see that the integration can be done by setting $t = \left(\frac{u}{1+u}\right)^{1/2}$. Under this change of variable, we have

$$\begin{aligned} A &= \pi \int_0^{2\sqrt{2}/3} \frac{1}{1-t^2}\,t\,\frac{2\,t}{(1-t^2)^2}\,\mathrm{d}t \\ &= 2\,\pi \int_0^{2\sqrt{2}/3} \frac{t^2}{(1-t^2)^3}\,\mathrm{d}t. \end{aligned} \tag{8}$$

Now from integration by parts of $\displaystyle\int \frac{\mathrm{d}t}{(1-t^2)^2}$ we have

$$\int \frac{t^2}{(1-t^2)^3}\,\mathrm{d}t = \frac{1}{4}\left[\frac{t}{(1-t^2)^2} - \int \frac{\mathrm{d}t}{(1-t^2)^2}\right]. \tag{9}$$

Now we apply partial fraction to calculate

$$\frac{1}{(1-t^2)^2} = \frac{A}{1+t} + \frac{B}{(1+t)^2} + \frac{C}{1-t} + \frac{D}{(1-t)^2} \Longrightarrow A = B = C = D = \frac{1}{4}. \tag{10}$$

$$\int \frac{\mathrm{d}t}{(1-t^2)^2} = \frac{1}{4}\left[\ln\left(\frac{1+t}{1-t}\right) + \frac{2\,t}{1-t^2}\right] + C. \tag{11}$$

$$A = \frac{\pi}{2}\left[\frac{t}{(1-t^2)^2} - \frac{1}{4}\left[\ln\left(\frac{1+t}{1-t}\right) + \frac{2\,t}{1-t^2}\right]\right]\Big|_{t=0}^{t=2\sqrt{2}/3} = \frac{51\sqrt{2}\,\pi}{2} - \frac{\pi}{2}\ln\left(1+\sqrt{2}\right). \tag{12}$$

Exercise 1. Alternatively, calculate A through $x = \tan\ t$ and through $A = \pi \int_0^8 \sqrt{u^2+u}\,\mathrm{d}u \overset{\sqrt{u^2+u}=u+t}{=\!=\!=\!=\!=\!=} \cdots$.