Math 118 Winter 2015 Lecture 47 (Apr. 9, 2015)

Final Review II: Curves and Surfaces

- Convexity.

Example 1. Let $f\colon \mathbb{R} \mapsto \mathbb{R}$ be convex.

i. Further assume that f is twice differentiable on $\mathbb{R}$. Prove that $e^{f(x)}$ is convex.

ii. Prove that $e^{f(x)}$ is convex without any further assumption.

iii. Find $f\colon \mathbb{R} \mapsto \mathbb{R}$ not convex but $e^{f(x)}$ is convex.

Proof.

i. We calculate

$$(e^f)'' = (e^f f')' = e^f f'' + e^f (f')^2 \geqslant 0 \tag{1}$$

as $f'' \geqslant 0$.

ii. First notice that as $(e^x)'' = e^x \geqslant 0$, e^x is convex. Now let $x, y \in \mathbb{R}, \lambda \in [0,1]$ be arbitrary. We have, by convexity of f and monotonicity of e^x, and then convexity of e^x,

$$e^{f(\lambda x + (1-\lambda) y)} \leqslant e^{\lambda f(x) + (1-\lambda) f(y)} \leqslant \lambda e^{f(x)} + (1-\lambda) e^{f(y)}. \tag{2}$$

iii. We prove that $f(x) = \begin{cases} 0 & x \leqslant 1 \\ \ln x & x > 1 \end{cases}$ is not convex. But $e^{f(x)} = \begin{cases} 1 & x \leqslant 1 \\ x & x > 1 \end{cases}$ is convex.

First as $(\ln x)'' = -\frac{1}{x^2} < 0$ for $x > 1$, we see that $f(x)$ is concave on $(1, \infty)$ and therefore is not convex on $\mathbb{R}$. Now consider $x_1, x_2 \in \mathbb{R}$ and $\lambda \in [0,1]$ arbitrary. There are three cases. Wlog $x_1 < x_2$.

1. $x_1, x_2 \leqslant 1$. Then $e^{f(x_1)} = e^{f(x_2)} = e^{f(\lambda x_1 + (1-\lambda) x_2)} = 1 \Longrightarrow e^{f(\lambda x_1 + (1-\lambda) x_2)} = \lambda e^{f(x_1)} + (1-\lambda) e^{f(x_2)}$;
2. $x_1, x_2 > 1$. We have again $e^{f(\lambda x_1 + (1-\lambda) x_2)} = \lambda e^{f(x_1)} + (1-\lambda) e^{f(x_2)}$.
3. $x_1 \leqslant 1 < x_2$. In this case we have $e^{f(\lambda x_1 + (1-\lambda) x_2)} \leqslant e^{f(\lambda + (1-\lambda) x_2)} = \lambda + (1-\lambda) x_2 = \lambda e^{f(x_1)} + (1-\lambda) e^{f(x_2)}$. $\square$

- Arc length.
  - $y = f(x), a \leqslant x \leqslant b$.
$$l = \int_a^b \sqrt{1 + f'(x)^2}\, \mathrm{d}x. \tag{3}$$
  - $x = x(t), y = y(t), a \leqslant t \leqslant b$.
$$l = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, \mathrm{d}t. \tag{4}$$
  - Polar coordinates $r = r(t), \theta = \theta(t)$, $a \leqslant t \leqslant b$.
$$l = \int_a^b \sqrt{r'(t)^2 + r(t)^2\, \theta'(t)^2}\, \mathrm{d}t. \tag{5}$$
  - Trivial generalization to arc length of spatial curves.

Example 2. Compute the arc length of the astroid $x = \cos^3 t$, $y = \sin^3 t$.

Solution. As $x(t)$ and $y(t)$ have period 2π, the curve is given by $0 \leqslant t \leqslant 2\pi$. We have

$$\begin{aligned} l &= \int_0^{2\pi} \sqrt{(3\cos^2 t \sin t)^2 + (3\sin^2 t \cos t)^2}\, \mathrm{d}t \\ &= \int_0^{2\pi} 3\,|\sin t \cos t|\, \mathrm{d}t \\ &= \frac{3}{2}\int_0^{2\pi} |\sin 2t|\, \mathrm{d}t \\ &= 6. \end{aligned} \tag{6}$$

- Area of plane regions.
  - $a \leqslant x \leqslant b$, $g(x) \leqslant y \leqslant f(x)$.

$$A = \int_a^b [f(x) - g(x)]\, \mathrm{d}x. \tag{7}$$

  - $x = x(t)$, $y = y(t)$, $a \leqslant t \leqslant b$. Closed curve. Counter-clockwise as t increases.

$$A = -\int_a^b y(t)\, x'(t)\, \mathrm{d}t = \int_a^b x(t)\, y'(t)\, \mathrm{d}t = \frac{1}{2}\int_a^b [x(t)\, y'(t) - y(t)\, x'(t)]\, \mathrm{d}t. \tag{8}$$

Example 3. Calculate the area bounded by the curve $x = a\cos t$, $y = b\sin t$, $a, b > 0$.

Solution. We have

$$A = \int_0^{2\pi} (a\cos t)\,(b\sin t)'\, \mathrm{d}t = \pi\, a\, b. \tag{9}$$

- Volume.
  - Area of cross-section with $x = x_0$: $A(x_0)$,

$$V = \int_a^b A(x)\, \mathrm{d}x. \tag{10}$$

Example 4. Find the volume of the torus generated by revolving a circle of radius 1 whose center is 2 away from the axis.

Solution. Let the axis be x-axis. Let the center of the circle be $(0, 2)$. The cross-section of the torus with the plane $x = c$ is 0 when $c > 1$ or $c < -1$. For $c \in [-1, 1]$ the cross-section is an anulus with inner radius $2 - \sqrt{1 - c^2}$ and outer radius $2 + \sqrt{1 - c^2}$. Thus we have

$$A(x) = \pi\left[\left(2 + \sqrt{1 - x^2}\right)^2 - \left(2 - \sqrt{1 - x^2}\right)^2\right] = 8\pi\sqrt{1 - x^2}. \tag{11}$$

The volume is now given by

$$8\pi\int_{-1}^1 \sqrt{1 - x^2}\, \mathrm{d}x \overset{x = \sin t}{=\!=\!=} 8\pi\int_{-\pi/2}^{\pi/2} \cos^2 t\, \mathrm{d}t = 4\pi^2. \tag{12}$$