Math 118 Winter 2015 Lecture 45 (Apr. 2, 2015)

Note. This lecture is based on Chapters 9 of *Proofs from the Book* by M. Aigner and G. M. Ziegler, 4th ed., Springer, 2010, and various online resources.

> **The problem.** specify "two tetrahedra of equal bases and equal altitudes which can in no way be split into congruent tetrahedra..."
>
> David Hilbert, 1900.

- The Wallace-Bolyai-Gerwien Theorem.

Theorem 1. (Wallace-Bolyai-Gerwien) *Consider two simple polygons in $\mathbb{R}^2$ with equal area. Then one can be cut into finitely many pieces and re-assemble to the other.*

Proof. We will only sketch the proof here. See www.cut-the-knot.org/do_you_know/Bolyai.shtml for more details as well as java animations. You can also search "Bolyai-Gerwien Theorem" on youtube to see animation of the steps below.

We prove that any simple polygon in $\mathbb{R}^2$ can be cut and re-assemble into a square of equal area in four steps.

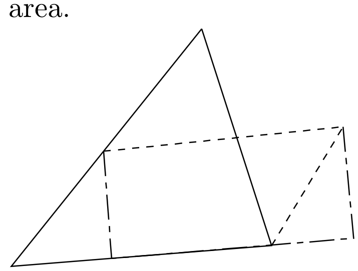
Exercise 1. Explain why this is sufficient to prove the theorem.

1. Any simple polygon in $\mathbb{R}^2$ can be cut into finitely many triangles.

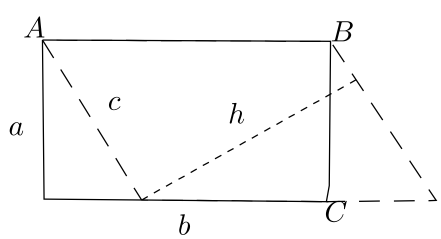
   This is a trivial statement that becomes not that trivial when you start to think about it. One way to prove this is to pick a direction that is not parallel to any of the sides of the polygon, and then cut with lines passing the vertices and in this direction.

   Exercise 2. Prove that each piece after such "cutting" is either a triangle or a trapezoid, which can be cut into two triangles.

2. Any triangle can be cut and re-assemble into first a parallelogram and then a rectangle of equal area.

3. Any rectangle can be cut and re-assemble into a square of equal area.

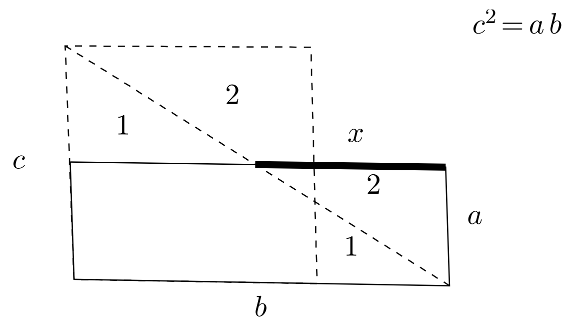
   

   $a\,b = c^2.$

   Exercise 3. Prove that $h = c$.

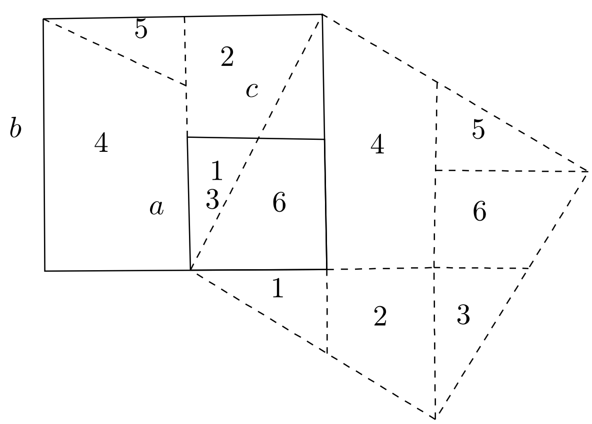
   Exercise 4. What if, in the above, the line h intersects AB instead of BC?

   Alternative proof.

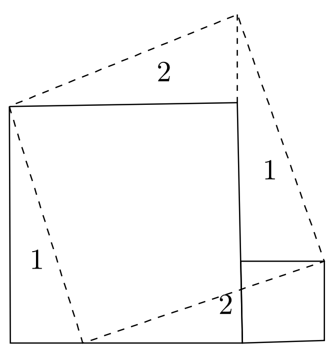


Exercise 5. Prove that $x = c$.

4. Any two squares can be cut and re-assemble into a square of area the sum of the areas of the original two squares.[1]



A simpler proof:



Note that in this proof the pieces are only translated during the re-assembly. ☐

- Tarski's "circle squaring" problem.

  The Problem. (ALFRED TARSKI 1925) Let B be the unit disk in $\mathbb{R}^2$ and let C be the square of area π. Is it possible to dissect B into finitely many pieces and re-assemble them into C?

1. In fact this step is not necessary, as from the previous step we already see that any rectangle can be cut and re-assemble into any other rectangle with equal area, as a easy consequence any finite collection of rectangles can be cut and re-assemble together in to a square. But it is cool to see a "direct" proof of Pythagorean Theorem.

Remark 2. The background of this problem is as follows. Around 1900 people are trying to understand measures – areas, volumes, etc. In 1924, Stefan Banach and Alfred Tarski showed that one can dissect the unit ball in $\mathbb{R}^3$ into finitely many pieces and then reassemble into two unit balls. On the other hand, around the same time Banach proved that such things cannot happen in 2D, where dissection-reassembly must leave areas unchanged. In light of the Wallace-Bolyai-Gerwien Theorem above, one naturally asks whether any two plane shapes of the same area can be dissected and reassembled into each other.

THEOREM 3. (M. LACZKOVICH 1990) *One can cut B into about 10^{50} different pieces and re-assemble them into C. Furthermore the pieces are only translated, not rotated, during the re-assembly.*

Remark 4. We should emphasize that Axiom of Choice is used in the proof and as a consequence each of the 10^{50} pieces is "non-measurable" in the Lebesgue sense. In particular, one cannot accomplish this using paper and scissors. See the post "What's wrong with this solution of Tarski's circle-squaring problem?" for a quite clever proof. In fact, it has been proved by Dubins, Hirsch, and Karush in 1963 that circle-squaring is not possible even if our scissors can "cut" along any continuous curve.

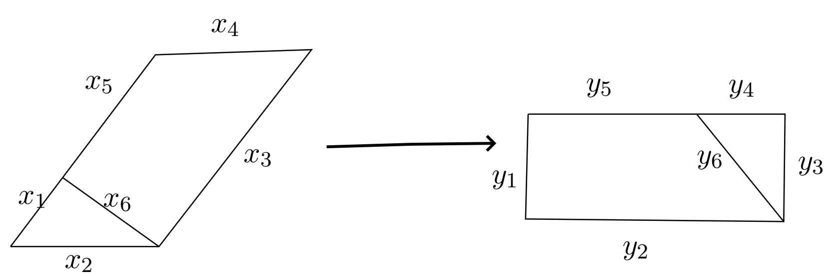
- (MAX DEHN 1902) A regular tetrahedron cannot be cut and re-assemble to a cube of same volume.

  Proof. [2]Assume the contrary. Call the tetrahedron T and the cube C. We assume that T can be cut into $T_1, \ldots, T_n$ and C into $C_1, \ldots, C_n$ such that T_i and C_i are congruent.

  - The Pearl Lemma.

    LEMMA 5. *It is possible to assigne a positive integer (number of "pearls") to each edge segment in the decompositions $T = \cup T_i$ and $C = \cup C_i$ such that each edge of P_i is assigned the same number of "pearls" as the corresponding edge of C_i.*

    Remark 6. This lemma in fact holds in any dimension. Below is a 2D illustration.



Here the claim is that there are positive integers $x_1, \ldots, x_6, y_1, \ldots, y_6$ satisfying

$$x_1 = y_4, \quad x_2 = y_6, \quad x_6 = y_3, \quad x_6 = y_1, \quad x_3 = y_2, \quad x_4 = y_6, \quad x_5 = y_5. \tag{1}$$

Proof. Assign a variable x_i to each segment of the decomposition of T and y_j to each segment of the decomposition of Q. Thus we have a system of equations

$$\sum \text{some } x_i' s - \sum \text{some } y_j' s = 0. \tag{2}$$

2. Note that this is not Dehn's original proof.

The goal is to prove that there are $x_i, y_j \in \mathbb{N}$ satisfying this system.

We observe that if we relax and condition and ask for $x_i, y_j \in \mathbb{R}, x_i, y_j > 0$, then the existence of such numbers is trivial: We just take x_i, y_j to be the length of the segments. The desired conclusion now follows from the following "cone lemma".

LEMMA 7. (CONE LEMMA) *If a system of homogeneous linear equations with integer coefficients has a positive real solution, then it also has a positive integer solution.*

Proof. The proof uses induction and is a bit technical, see Chapters 9 of *Proofs from the Book*. Here we make a few observations so that the claim feels less surprising.

i. (This is the most important observation!) If a system of homogeneous linear equations has a positive rational solution, then it also has a positive integer solution.

ii. As rational numbers are dense, it may be possible to "modify" the original positive real solution a bit to make it rational.

iii. When the solution is unique of course such "modification" is not possible. However in this case this unique solution has to be 0. □ □

- Bricard's condition.

THEOREM 8. *Let P, Q be 3D polyhedra that can be cut and reassemble to each other, and let $\alpha_1, \ldots, \alpha_r$ and $\beta_1, \ldots, \beta_s$ be the dihedral angles of P, Q respectively. Then there are positive integers $m_1, \ldots, m_r$, $n_1, \ldots, n_s$ and $k \in \mathbb{Z}$ such that*

$$m_1 \alpha_1 + \cdots + m_r \alpha_r = n_1 \beta_1 + \cdots + n_s \beta_s + k \pi. \tag{3}$$

Proof. Let Σ_1 be the sum of all dihedral angles at each pearl for every P_i and Σ_2 be the sum of that for every Q_j. Then $\Sigma_1 = \Sigma_2$ as both are the same weighted sum of all dihedral angles for all pieces. On the other hand we can first add the angles at each pearl and then add the sum together. This gives

$$\Sigma_1 = m_1 \alpha_1 + \cdots + m_r \alpha_r + k_1 \pi, \quad \Sigma_2 = n_1 \beta_1 + \cdots + n_s \beta_s + k_2 \pi \tag{4}$$

for some positive integers $m_1, \ldots, m_r$, $n_1, \ldots, n_s$ and $k_1, k_2 \in \mathbb{Z}$. The conclusion now follows. □

- Finishing the proof.

For a regular tetrahedron all $\alpha_i = \arccos\frac{1}{3}$ and for a cube all $\beta_j = \frac{\pi}{2}$. Therefore $m \arccos\frac{1}{3} = n\,\frac{\pi}{2} + k\,\pi \Longrightarrow \frac{\arccos(1/3)}{\pi} \in \mathbb{Q}$.

PROPOSITION 9. *Let $n \geqslant 3$ be odd. Then $\frac{1}{\pi} \arccos\left(\frac{1}{\sqrt{n}}\right)$ is irrational.*

Proof. Denote $\varphi_n := \arccos(1/\sqrt{n})$.

Exercise 6. Prove that $\cos(k\,\varphi_n) = \frac{A_k}{(\sqrt{n})^k}$ for some integer A_k not divisible by n. (Hint:[3])

If $\varphi_n = \frac{l}{k}\pi$ for some $l, k \in \mathbb{Z}$, $\cos(k\,\varphi_n) = \pm 1 \Longrightarrow |A_k| = n^{k/2}$. Contradiction. □

Thus ends the proof. □

3. $\cos((k+1)\,\varphi_n) + \cos((k-1)\,\varphi_n)$.