Math 118 Winter 2015 Lecture 44 (Apr. 1, 2015)

Note. This lecture is based on Chapters 2, 4, 6 of *Irrational Numbers* by Ivan Niven, The Mathematical Association of America, 1956.

- Irrationality of certain numbers.
  - Irrationality of π.

    Theorem 1. *π^2 is irrational.*

    Proof. Assume $\pi^2 = \frac{a}{b}$ for some $a, b \in \mathbb{Z}, (a, b) = 1, b > 0$. Define

    $$f(x) = \frac{x^n (1-x)^n}{n!}. \tag{1}$$

    Exercise 1. Prove that for every $j \in \mathbb{N} \cup \{0\}$, $f^{(j)}(0), f^{(j)}(1) \in \mathbb{Z}$.

    Now define

    $$F(x) := b^n \left[\pi^{2n} f(x) - \pi^{2n-2} f''(x) + \pi^{2n-4} f^{(4)}(x) - \cdots + (-1)^n f^{(2n)}(x) \right]. \tag{2}$$

    Exercise 2. Prove that $F(0), F(1) \in \mathbb{Z}$.

    Exercise 3. Prove that

    $$\frac{\mathrm{d}}{\mathrm{d}x}[F'(x) \sin(\pi x) - \pi F(x) \cos(\pi x)] = \pi^2 a^n f(x) \sin(\pi x). \tag{3}$$

    Therefore

    $$\pi a^n \int_0^1 f(x) \sin(\pi x)\, \mathrm{d}x = F(1) + F(0). \tag{4}$$

    Now notice that $0 < f(x) \sin(\pi x) < \frac{1}{n!}$ for $x \in (0, 1)$. Therefore

    $$0 < \pi a^n \int_0^1 f(x) \sin(\pi x)\, \mathrm{d}x < \frac{\pi a^n}{n!}. \tag{5}$$

    Exercise 4. Prove that there is $n_0 \in \mathbb{N}$ such that

    $$0 < \pi a^n \int_0^1 f(x) \sin(\pi x)\, \mathrm{d}x < 1. \tag{6}$$

    This gives a contradiction as $F(0) + F(1) \in \mathbb{Z}$. $\square$

  - Irrationality of certain numbers.

    Theorem 2. *For any rational number $r \neq 0$, $\cos r$ is irrational.*

    Proof. Wlog $r = \frac{a}{b}$ where $a, b \in \mathbb{N}$. We assume $\cos r = \frac{d}{k}$ for $d, k \in \mathbb{Z}$.

    Now define

    $$f(x) = \frac{x^{p-1} (a - b x)^{2p} (2a - b x)^{p-1}}{(p-1)!} = \frac{(r - x)^{2p} [r^2 - (r - x)^2]^{p-1} b^{3p-1}}{(p-1)!} \tag{7}$$

    where p is some odd prime greater than a.

    Exercise 5. Prove that $f^{(j)}(0) \in \mathbb{Z}$ for every $j \in \mathbb{N} \cup \{0\}$.

    Now define

    $$F(x) = f(x) - f''(x) + f^{(4)}(x) - \cdots - f^{(4p-2)}(x). \tag{8}$$

Exercise 6. Prove that $F'(r)=0$.

Exercise 7. Prove that

$$\int_0^r f(x)\sin x\,\mathrm{d}x = F(0) - F(r)\cos r. \tag{9}$$

Exercise 8. Prove that $f^{(j)}(0)$ is a multiple of p unless $j=p-1$, and that $f^{(p-1)}(0) = a^{2p}(2a)^{p-1}$.

Now as $p>a$, we see that $F(0)=q$ co-prime with p.
Next we study $F(r)$. By (7) we easily see that

$$f(r-x) = \frac{x^{2p}(r^2-x^2)^{p-1}b^{3p-1}}{(p-1)!} = \frac{x^{2p}(a^2-b^2x^2)^{p-1}b^{p+1}}{(p-1)!}. \tag{10}$$

Exercise 9. Prove that $f^{(j)}(r)$ is divisible by p for every $j\in\mathbb{N}\cup\{0\}$ and thus $p|F(r)$.

Therefore $F(r)=p\,m$ for some $m\in\mathbb{Z}$. As $\cos r=\frac{d}{k}$ we have

$$k\int_0^r f(x)\sin x\,\mathrm{d}x = k\,q - p\,m\,d. \tag{11}$$

Exercise 10. Show that for large enough p,

$$\left|k\int_0^r f(x)\sin x\,\mathrm{d}x\right| < 1, \tag{12}$$

and conclude that $k\,q-p\,m\,d=0$.

Exercise 11. Prove that there is a contradiction now. □

Exercise 12. Prove $\pi\notin\mathbb{Q}$ in one sentence using Theorem 2.

Exercise 13. Prove the following.

- For any rational number $r\neq 0$, $\sin r$ is irrational.
- For any rational number $r\neq 0$, $\tan r$ is irrational.

(Hint:[1])

Problem 1. For any rational number $r\neq 0$, $\cosh r=\frac{e^r+e^{-r}}{2}$ is irrational. (Hint:[2])

- Properties of irrational numbers.
  - Approximation by rational numbers.

THEOREM 3. *Let $\alpha\in\mathbb{R}$ be irrational. Then there are infinitely many rationals $\frac{h}{k}$ such that $\left|\alpha-\frac{h}{k}\right|<\frac{1}{k^2}$.*

Proof. Wlog assume $\alpha>0$. Let $\{x\}$ denote the fractional part of x, for example $\{\pi\}=0.1415926\ldots$

Now let $n\in\mathbb{N}$ be arbitrary. Consider the $n+1$ numbers

$$0,\{\alpha\},\{2\,\alpha\},\ldots,\{n\,\alpha\}\in[0,1). \tag{13}$$

If we divide $[0,1)$ into n intervals $\left[0,\frac{1}{n}\right),\left[\frac{1}{n},\frac{2}{n}\right),\ldots,\left[\frac{n-1}{n},1\right)$, we see that two of the $n+1$ numbers must fall in the same interval. Thus there are $k_1,k_2\in\{0,1,\ldots,n\}$ such that $|\{k_1\,\alpha\}-\{k_2\,\alpha\}|<\frac{1}{n}$. Let $k:=|k_2-k_1|$.

1. $\cos 2\,r$.
2. $F(x)=f(x)+f''(x)+f^{(4)}(x)+\cdots+f^{(4p-2)}(x)$.

Exercise 14. Prove that there is $h\in\mathbb{Z}$ such that $|k\,\alpha-h|<\frac{1}{n}$ and this gives $\left|\alpha-\frac{h}{k}\right|<\frac{1}{n\,k}\leqslant\frac{1}{k^2}$.

Thus we have shown that for every $n\in\mathbb{N}$, there is $k\in\{1,2,\ldots,n\}$ and $h\in\mathbb{Z}$ such that $\left|\alpha-\frac{h}{k}\right|<\frac{1}{n\,k}\leqslant\frac{1}{k^2}$. Denote by k_n the largest of such k for a given n.

Exercise 15. Prove that $\lim_{n\to\infty}k_n=\infty$ and therefore there are infinitely many $\frac{h}{k}$ satisfying $\left|\alpha-\frac{h}{k}\right|<\frac{1}{k^2}$. (Hint:[3]) □

Exercise 16. Let $r\in\mathbb{Q}$. Prove that there is $b\in\mathbb{N}$ such that $\left|r-\frac{h}{k}\right|\geqslant\frac{1}{b\,k}$ for all rationals $\frac{h}{k}$ unless $\frac{h}{k}=r$.

- The "most irrational" number.

Remark 4. Through application of the theory of continued fractions, Theorem 3 can be improved as follows.

THEOREM. *Let α be irrational. Then there are infinitely many rational numbers $\frac{h}{k}$ such that*

$$\left|\alpha-\frac{h}{k}\right|<\frac{1}{\sqrt{5}\,k^2}. \tag{14}$$

Proof. See Chapter 6 of *Irrational Numbers* by Ivan Niven. □

THEOREM 5. *Let $\alpha=\frac{1+\sqrt{5}}{2}$ and $c>\sqrt{5}$. Then there are only finitely many rational numbers $\frac{h}{k}$ (note that we assume h,k to be co-prime) such that*

$$\left|\alpha-\frac{h}{k}\right|<\frac{1}{c\,k^2}. \tag{15}$$

Proof. Let's see what are the restrictions for $\left|\alpha-\frac{h}{k}\right|<\frac{1}{c\,k^2}$. Write $\frac{\sqrt{5}+1}{2}-\frac{h}{k}=\frac{1}{x\,k^2}$. Then $|x|>c>\sqrt{5}$. Rearranging, we have

$$\frac{1}{x\,k}-\frac{\sqrt{5}\,k}{2}=\frac{k}{2}-h. \tag{16}$$

Squaring and simplifying, we have

$$\frac{1}{x^2\,k^2}-\frac{\sqrt{5}}{x}=h^2-h\,k-k^2. \tag{17}$$

Now we check

$$\left|\frac{1}{x^2\,k^2}-\frac{\sqrt{5}}{x}\right|<\frac{1}{k^2}+\frac{\sqrt{5}}{c}. \tag{18}$$

Thus there is $k_0\in\mathbb{N}$ such that $k>k_0\Longrightarrow\frac{1}{k^2}+\frac{\sqrt{5}}{c}<1$. As $h^2-h\,k-k^2\in\mathbb{Z}$, if $k>k_0$ there must hold $h^2-h\,k-k^2=0$. But this is not possible as $(h,k)=1$. Therefore $\left|\frac{1+\sqrt{5}}{2}-\frac{h}{k}\right|<\frac{1}{c\,k^2}$ implies $k\leqslant k_0$ and the proof ends. □

Remark 6. The following theorem earned Klaus Roth (1925 –) a Fields Medal in 1958.

3. Assume the contrary. Let $K=\max k_n$. Show that $|K\,\alpha-h|<\frac{1}{n}$ cannot hold as $n\longrightarrow\infty$.

THEOREM 7. *Let $\alpha \in \mathbb{R}$. If there are $s > 2$ and infinitely many rationals $\frac{h}{k}$ such that*

$$\left| \alpha - \frac{h}{k} \right| < \frac{1}{k^s}, \tag{19}$$

then α is transcendental.

Note that a number α is transcendental if $a_n \alpha^n + \cdots + a_1 \alpha + a_0 \neq 0$ for any $n \in \mathbb{N}$ and $a_n, \ldots, a_0 \in \mathbb{Q}$.

- Ergodicity.

DEFINITION 8. *We say a sequence of numbers $\alpha_1, \alpha_2, \ldots \in [0, 1]$ is "uniformly distributed" in $[0, 1]$ if and only if for every $I = [a, b] \subseteq [0, 1]$, there holds*

$$\lim_{n \to \infty} \frac{n(I)}{n} = b - a \tag{20}$$

where $n(I) =$ number of $\alpha_1, \ldots, \alpha_n$ that lie in $[a, b]$.

Exercise 17. Let r be rational. Prove that $\{r\}, \{2 r\}, \ldots$ is not uniformly distributed in $[0, 1]$.

THEOREM 9. *Let α be irrational. Then $\{\alpha\}, \{2 \alpha\}, \ldots$ is uniformly distributed in $[0, 1]$.*

Proof. Let's first assume the following result:

THEOREM 10. *$\{\beta_n\} \subset [0, 1]$ is uniformly distributed if*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \cos(2 \pi m \beta_j) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \sin(2 \pi m \beta_j) = 0 \tag{21}$$

for every $m \in \mathbb{N}$.

Assuming Theorem 10 now, the uniform distribution of $\{n \alpha\}$ becomes obvious.

Exercise 18. Prove that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \cos(2 \pi m j \alpha) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \sin(2 \pi m j \alpha) = 0 \tag{22}$$

(Hint:[4]) □

Proof. (OF THEOREM 10) We will give a "pseudo-proof" here to illustrate the main idea. For the true proof, see §6.4 of *Irrational Numbers* by Ivan Niven.

Let $[a, b] \subseteq [0, 1]$ be arbitrary. We try to prove (20) assuming (21). Define the function $g(x) := \begin{cases} 1 & x \in [a, b] \\ 0 & x \notin [a, b] \end{cases}$. Pretend[5] that the following holds uniformly on $[0, 1]$:

$$g(x) = a_0 + \sum_{m=1}^{\infty} [a_m \cos(2 \pi m x) + b_m \sin(2 \pi m x)]. \tag{23}$$

Exercise 19. Prove that $a_0 = b - a$.

Exercise 20. Prove that $|a_m|, |b_m| \leqslant 2$ for all $m \in \mathbb{N}$. (Hint:[6])

4. Review our proof for convergence of $\sum_{n=1}^{\infty} \frac{\sin(n x)}{n}$.

5. (23) does not hold uniformly for $x \in [0, 1]$. But this can be fixed through some technical approximation argument.

We observe that

$$n([a,b]) = \sum_{j=1}^{n} g(\beta_j). \tag{24}$$

Now let $\varepsilon > 0$ be arbitrary. By the uniform convergence of (23) there is M_1 such that

$$\forall x \in [0,1], \qquad \left| \sum_{m=M_1}^{\infty} [a_m \cos(2\pi m x) + b_m \sin(2\pi m x)] \right| < \frac{\varepsilon}{2}. \tag{25}$$

On the other hand, by assumption (20) there is N_1 such that

$$\forall n > N_1, \forall m \leqslant M_1, \qquad \left| \frac{1}{n} \sum_{j=1}^{n} \cos(2\pi m \beta_j) \right|, \left| \frac{1}{n} \sum_{j=1}^{n} \sin(2\pi m \beta_j) \right| < \frac{\varepsilon}{8 M_1}. \tag{26}$$

Now we calculate, for $n > \max\{N_1, M_1\}$,

$$\begin{aligned} \frac{n([a,b])}{n} &= \frac{1}{n} \sum_{j=1}^{n} g(\beta_j) \\ &= (b-a) + \frac{1}{n} \sum_{j=1}^{n} \left[\sum_{m=M_1+1}^{\infty} [a_m \cos(2\pi m \beta_j) + b_m \sin(2\pi m \beta_j)] \right] \\ &\quad + \frac{1}{n} \sum_{j=1}^{n} \left[\sum_{m=1}^{M_1} [a_m \cos(2\pi m \beta_j) + b_m \sin(2\pi m \beta_j)] \right]. \end{aligned} \tag{27}$$

Now clearly

$$\left| \frac{1}{n} \sum_{j=1}^{n} \left[\sum_{m=M_1+1}^{\infty} [a_m \cos(2\pi m \beta_j) + b_m \sin(2\pi m \beta_j)] \right] \right| < \frac{\varepsilon}{2}. \tag{28}$$

On the other hand, we can switch the order of summation in the first term to obtain

$$\left| \frac{1}{n} \sum_{j=1}^{n} \left[\sum_{m=1}^{M_1} a_m \cos(2\pi m \beta_j) \right] \right| = \left| \sum_{m=1}^{M_1} a_m \left[\frac{1}{n} \sum_{j=1}^{n} \cos(2\pi m \beta_j) \right] \right| < \frac{\varepsilon}{4}. \tag{29}$$

Similarly

$$\left| \frac{1}{n} \sum_{j=1}^{n} \left[\sum_{m=1}^{M_1} b_m \sin(2\pi m \beta_j) \right] \right| < \frac{\varepsilon}{4}. \tag{30}$$

Thus we have, for $n > \max\{N_1, M_1\}$, $\left| \frac{n([a,b])}{n} - (b-a) \right| < \varepsilon$ and the conclusion follows. □

Problem 2. (Ergodicity) Let $\alpha \in \mathbb{Q}^c$. Let $f(x)$ be continuous on $[0,1]$. Prove

$$\lim_{n \longrightarrow \infty} \frac{f(\alpha) + f(2\alpha) + \cdots + f(n\alpha)}{n} = \int_0^1 f(x)\, \mathrm{d}x. \tag{31}$$

Show that (31) does not hold if $\alpha \in \mathbb{Q}$. Does (31) still hold if we only assume $f(x)$ to be Riemann integrable on $[0,1]$?

6. Multiply (23) by $\cos(2\pi m x)$ (or $\sin(2\pi m x)$) and then integrate.