MATH 118 WINTER 2015 LECTURE 42 (MAR. 27, 2015)

Note. Please make sure you read Chapter 8 of Dr. Bowman's book.

- Calculation of volume.

  If the area of the cross-section of the 3D shape and the plane parellel to y-z plane, passing $(x, 0, 0)$ is $A(x)$, then the volume of the shape between $x = a$ and $x = b$ is given by

$$\int_a^b A(x)\,\mathrm{d}x. \tag{1}$$

- Volume of solids of revolution.

  Consider the graph of a function $y = f(x)$ on $a \leqslant x \leqslant b$ and the following two situations. For simplicity of discussion we assume $a \geqslant 0$, $f(x) \geqslant 0$. The generalization to other situations is easy to do once one understands this case.

  i. We rotate the region $\{(x, y) |\, a \leqslant x \leqslant b, 0 \leqslant y \leqslant f(x)\}$ around the x-axis; (In this case we assume $f(x) \geqslant 0$).

     We cut the shape by planes parallel to the y-z plane. Each cross-section is a disk with radius $f(x)$. Therefore the volume is given by

$$V = \int_a^b \pi\, f(x)^2\,\mathrm{d}x. \tag{2}$$

  ii. We rotate the region $\{(x, y) |\, a \leqslant x \leqslant b, 0 \leqslant y \leqslant f(x)\}$ around the y-axis. (In this case we assume $a \geqslant 0$, $f(x) \geqslant 0$).

     In this case cutting by planes is not efficient. Instead we cut by cylinders. Each cut is a cylinder with radius x and height $f(x)$, thus having area $2\,\pi\,x\,f(x)$. Therefore the volume is given by

$$V = 2\,\pi \int_a^b x\, f(x)\,\mathrm{d}x. \tag{3}$$

  Remark 1. Of course, if we rotate two graphs $f(x) \geqslant g(x)$ around the y axis, the volume would be

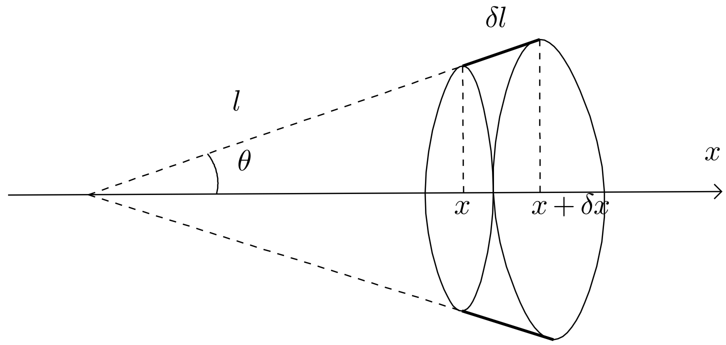
$$V = 2\,\pi \int_a^b x\,[f(x) - g(x)]\,\mathrm{d}x. \tag{4}$$

  Exercise 1. Calculate the volume of the last example in yesterday's lecture: volume enclosed by $2\,z = x^2 + y^2$ and $x^2 + y^2 + z^2 = 3$, using (4).

- Surface area of surfaces of revolution.

  - Rotation of a graph $y = f(x), a \leqslant x \leqslant b$ around the x-axis.

    We replace the part of graph from x to $x + \delta x$ by a straight line segment connecting $f(x)$ and $f(x + \delta x) \approx f(x) + f'(x)\,\delta x$. If we denote $\theta = \arctan f'(x)$, then the surface formed by rotating this line segment around x-axis is the difference of two cones:



Recalling the formula for surface area of cones, we have the area to be

$$\begin{aligned} \delta A &= \pi\,(l+\delta l)\,f(x+\delta x) - \pi\,l\,f(x) \\ &\approx \pi\,[f(x)\,\delta l + l\,f'(x)\,\delta x] \\ &= 2\,\pi\,\frac{f(x)\,f'(x)}{\sin\theta}\,\delta x. \end{aligned} \tag{5}$$

Now we easily see that $\sin\theta = \frac{f'(x)}{\sqrt{1+f'(x)^2}}$, therefore we have

$$\delta A \approx 2\,\pi\,f(x)\,\sqrt{1+f'(x)^2}\,\delta x \tag{6}$$

and consequently the formula for the surface area should be

$$A = 2\,\pi \int_a^b f(x)\,\sqrt{1+f'(x)^2}\,\mathrm{d}x. \tag{7}$$

Remark 2. The discussion here is far from rigorous. On the other hand, the rigorous definition of surface area, in contrast to that of arc length, is subtle. Check out “Schwarz lantern”[1] to see why.

Example 3. We calculate the surface area of the unit sphere. The unit sphere can be seen as the surface formed by rotating $y=\sqrt{1-x^2}$ around the x-axis. Thus we have

$$A = 2\,\pi \int_{-1}^{1} \sqrt{1-x^2}\,\sqrt{1+[(\sqrt{1-x^2})']^2}\,\mathrm{d}x = 4\,\pi. \tag{8}$$

Example 4. (Torricelli’s Trumpet/Gabriel’s Horn) See https://magimathics.wordpress.com/2010/01/18/the-horn-of-gabriel for the historical significance of this example.

Consider the graph $y=\frac{1}{x}$, $x \geqslant 1$. We calculate the volume of its solid of revolution and the area of its surface of revolution.

– Volume.

$$V = \pi \int_1^{\infty} \left(\frac{1}{x}\right)^2 \mathrm{d}x = \pi. \tag{9}$$

– Surface area.

$$A = 2\,\pi \int_1^{\infty} \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,\mathrm{d}x > 2\,\pi \int_1^{\infty} \frac{\mathrm{d}x}{x} = +\infty. \tag{10}$$

1. For example at http://www.cut-the-knot.org/Curriculum/Calculus/SchwarzLantern.shtml.

Exercise 2. Apply Chebyshev's theorem to see whether the indefinite integral

$$\int \frac{1}{x} \sqrt{1 + \frac{1}{x^4}}\, \mathrm{d}x \tag{11}$$

is elementary. Calculate it if it is elementary.

Remark 5. Surface area of rotating a parametrized curve $(x(t), y(t)), a \leqslant t \leqslant b$ around the x-axis. We notice that in formula (7) can be viewed as the following:

$$A = 2\pi \int_a^b [\text{distance to the } x\text{-axis}] \cdot [\text{infinitesimal arc length}] \tag{12}$$

This suggests that the formula for the surface area of rotating a parametrized curve is

$$A = 2\pi \int_a^b y(t) \sqrt{x'(t)^2 + y'(t)^2}\, \mathrm{d}t. \tag{13}$$

Exercise 3. Calculate the surface area of the surface of revolution formed by rotating the cycloid $x = t - \sin t$, $y = 1 - \cos t$, $0 \leqslant t \leqslant 2\pi$ around the x-axis. (Ans:[2])

2. $64\pi/3$.