Math 118 Winter 2015 Lecture 40 (Mar. 25, 2015)

- Arc length of a graph.
  - Graph $y=f(x), a\leqslant x\leqslant b$:
$$l=\int_a^b \sqrt{1+f'(x)^2}\,\mathrm{d}x. \tag{1}$$
  - Parametrized curve $(x(t), y(t)), a\leqslant t\leqslant b$:
$$l=\int_a^b \sqrt{x'(t)^2+y'(t)^2}\,\mathrm{d}t. \tag{2}$$
  - Parametrized curve in polar coordinates $(r(t), \theta(t)), a\leqslant t\leqslant b$:
$$l=\int_a^b \sqrt{r'(t)^2+r(t)^2\,\theta'(t)^2}\,\mathrm{d}t. \tag{3}$$
    In particular when the curve is given by $r=r(\theta), a\leqslant\theta\leqslant b$,
$$l=\int_a^b \sqrt{r'(\theta)^2+r(\theta)^2}\,\mathrm{d}\theta. \tag{4}$$

Example 1. Cycloid. See en.wikipedia.org/wiki/Cycloid, mathworld.wolfram.com/Cycloid.html for background on this curve. The parametrized representation is
$$x(t)=t-\sin t, y(t)=1-\cos t, \qquad 0\leqslant t\leqslant 2\pi. \tag{5}$$
Thus we have
$$\begin{aligned} l &= \int_0^{2\pi} \sqrt{x'(t)^2+y'(t)^2}\,\mathrm{d}t \\ &= \sqrt{2}\int_0^{2\pi} \sqrt{1-\cos t}\,\mathrm{d}t \\ &= 2\int_0^{2\pi} \sqrt{\sin^2\frac{t}{2}}\,\mathrm{d}t \\ &= 2\int_0^{2\pi} \sin\frac{t}{2}\,\mathrm{d}t=8. \end{aligned} \tag{6}$$

Exercise 1. Explain why $\sqrt{\sin^2(t/2)}=\sin(t/2)$ in the above calculation.

- Area of plane regions.

Note. Since we do not discuss any measure theory, the discussion of area can only be semi-rigorous. Rigorous treatment will be done in multi-variable calculus.

  - Area between graphs.
    The simplest situation is to calculate the area of the region
$$a\leqslant x\leqslant b, \qquad g(x)\leqslant y\leqslant f(x). \tag{7}$$
    Clearly the area should be
$$A=\int_a^b [f(x)-g(x)]\,\mathrm{d}x. \tag{8}$$

Example 2. Calculate the area of the ellipsis $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution. We will treat the upper half as f and the lower half as g. Therefore

$$f(x) = b\sqrt{1 - \frac{x^2}{a^2}}, \qquad g(x) = -b\sqrt{1 - \frac{x^2}{a^2}}, \qquad -a \leqslant x \leqslant a, \tag{9}$$

and

$$A = 2\,b \int_{-a}^{a} \sqrt{1 - \frac{x^2}{a^2}}\,\mathrm{d}x. \tag{10}$$

Exercise 2. Prove that the area is given by $\pi\,a\,b$.

Exercise 3. Calculate the area enclosed by $y = e^x, y = e^{-x}, x = 0, x = 1$.

- Area between $x = \psi(y)$ and $x = \varphi(y)$.

The second simplest situation is the area of a region enclosed by $y = c$, $y = d$, $x = \psi(y), x = \varphi(y)$. Here we assume $c < d$, $\psi(y) \leqslant \varphi(y)$ for all $y \in [c, d]$. Then we have

$$A = \int_c^d [\varphi(y) - \psi(y)]\,\mathrm{d}y. \tag{11}$$

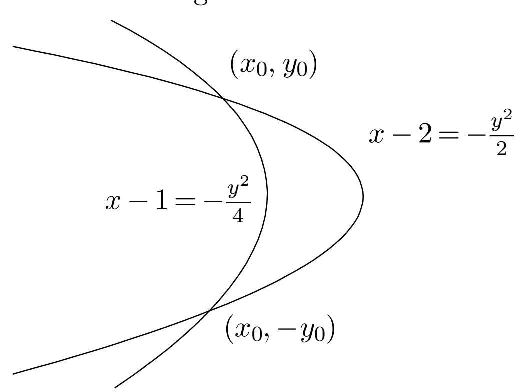
Example 3. Calculate the area between $y^2 = -4\,(x-1)$ and $y^2 = -2\,(x-2)$.

Solution. Note that we need to figure out c, d and which is ψ, which is φ. To do this we need some basic understanding of the shape of this region.

We write the two curves as

$$x - 1 = -\frac{y^2}{4}, \qquad x - 2 = -\frac{y^2}{2}. \tag{12}$$

We see that both are parabolas facing (opening to) left, with the second parabola to the right of the first one at the base.



The area is thus given by

$$\int_{-y_0}^{y_0} \left[\left(2 - \frac{y^2}{2}\right) - \left(1 - \frac{y^2}{4}\right)\right] \mathrm{d}y. \tag{13}$$

Thus all we need is calculating y_0. (x_0, y_0) satisfy

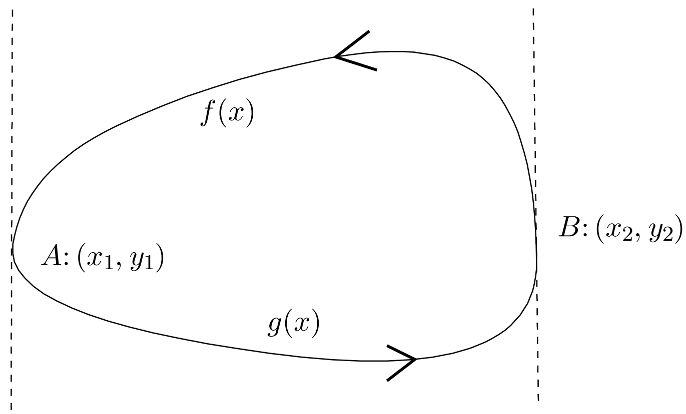
$$x_0 - 1 = -\frac{y_0^2}{4}, \qquad x_0 - 2 = -\frac{y_0^2}{2}. \tag{14}$$

From this we easily solve $x_0 = 0$, $y_0 = 2$.

- Area enclosed by a general parametrized curve $(x(t), y(t))$, $a \leqslant t \leqslant b$, $x(a) = x(b)$, $y(a) = y(b)$.

Exercise 4. Explain why we need $x(a) = x(b), y(a) = y(b)$.

Here we only consider curves enclosing a convex shape, although the formulas derived will apply to any closed curve.



We assume A is the point corresponding to $t = a$, b while B is the point corresponding to $t = c \in (a, b)$. We also assume that when t increases from a to c, $x(t)$ is strictly increasing with $(x(t), y(t))$ tracing the lower half of the curve; when t increases from c to b, $x(t)$ is strictly decreasing with $(x(t), y(t))$ tracing the upper half of the curve. In particular, when t increases from a to b, $(x(t), y(t))$ traces the curve counter-clockwisely.

Now if we can find functions $f(x), g(x)$ such that the upper/lower halves are their graphs, the area can be calculated as $\int_{x_1}^{x_2} [f(x) - g(x)]\, \mathrm{d}x$.

First consider $f(x)$. As $x(t)$ is strictly decreasing from x_2 to x_1 when t is increasing from c to b, there is an inverse function $t = T_f(x)$ on $[x_1, x_2]$. Thus we have $f(x) = y(T_f(x))$. Now we calculate

$$\begin{aligned} \int_{x_1}^{x_2} f(x)\, \mathrm{d}x &= \int_{x(b)}^{x(c)} y(T_f(x))\, \mathrm{d}x \\ &\overset{x=x(t)}{=\!=} \int_b^c y(t)\, x'(t)\, \mathrm{d}t \\ &= -\int_c^b y(t)\, x'(t)\, \mathrm{d}t. \end{aligned} \tag{15}$$

Exercise 5. Prove that

$$\int_{x_1}^{x_2} g(x)\, \mathrm{d}x = \int_a^c y(t)\, x'(t)\, \mathrm{d}t. \tag{16}$$

Thus we have

$$A = -\int_a^b y(t)\, x'(t)\, \mathrm{d}t. \tag{17}$$

Exercise 6. Show that if we parametrize the curve in the opposite (that is, clockwise) direction, we would have $-A = -\int_a^b y(t)\, x'(t)\, \mathrm{d}t$.

Exercise 7. Prove that

$$A = \int_a^b x(t)\, y'(t)\, \mathrm{d}t \tag{18}$$

and there also holds

$$A = \frac{1}{2} \int_a^b [x(t)\, y'(t) - y(t)\, x'(t)]\, \mathrm{d}t. \tag{19}$$

Example 4. Calculate the area enclosed by the cycloid and the x-axis.

Solution. We first represent the boundary of the region by one parametrized curve:

$$\tilde{x}(t) = \begin{cases} t - \sin t & 0 \leqslant t \leqslant 2\pi \\ 4\pi - t & 2\pi < t \leqslant 4\pi \end{cases}, \qquad \tilde{y}(t) = \begin{cases} 1 - \cos t & 0 \leqslant t \leqslant 2\pi \\ 0 & 2\pi < t \leqslant 4\pi \end{cases}. \tag{20}$$

Now we easily calculate

$$-\int_0^{4\pi} \tilde{y}(t)\, \tilde{x}'(t)\, \mathrm{d}t = -\int_0^{2\pi} (1 - \cos t)^2\, \mathrm{d}t = -3\pi. \tag{21}$$

Note that our parametrization $(\tilde{x}(t),\ \tilde{y}(t))$ traces the boundary of the region clockwisely, therefore the area is $-(-3\pi) = 3\pi$.