MATH 118 WINTER 2015 LECTURE 39 (MAR. 23, 2015)

- Arc length of a graph.

  Consider the graph of $y = f(x), a \leqslant x \leqslant b$. We try to establish a formula for the arc length l of this curve.

  Let $P: a = x_0 < x_1 < \cdots < x_n = b$ be an arbitrary partition of $[a, b]$. We connect the points $(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n))$ be straight line segments. The resulting polygonal curve has length

$$l(P) = \sum_{k=1}^{n} \sqrt{(x_k - x_{k-1})^2 + (f(x_k) - f(x_{k-1}))^2}. \tag{1}$$

Intuitively, we should accept the following:

i. $l(P) \leqslant l$. (A curve connecting two points is no shorter than the straight line connecting the same two points)

ii. As P gets finer and finer (that is having more and more points), $l(P)$ approaches l.

Thus the following definition is reasonable:

DEFINITION 1. *The arc length of the graph of $y = f(x), a \leqslant x \leqslant b$ is defined as*

$$l := \sup_P l(P). \tag{2}$$

THEOREM 2. *Under the following assumptions on f,*

i. f is continuous on $[a, b]$;

ii. f' is continuous on (a, b);

iii. $\lim_{x \to a+} f'$ and $\lim_{x \to b-} f'$ exist and are finite.

there holds

$$l = \int_a^b \sqrt{1 + f'(x)^2} \, \mathrm{d}x. \tag{3}$$

Exercise 1. Is the continuity of f on $[a, b]$ the consequence of the other two assumptions (on f')? Justify your claim.

Proof. Let P be an arbitrary partion of $[a, b]$. Then by MVT we have

$$l(P) = \sum_{k=1}^{n} \sqrt{1 + \left(\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}\right)^2} (x_k - x_{k-1}) = \sum_{k=1}^{n} \sqrt{1 + f'(c_k)^2} (x_k - x_{k-1}) \tag{4}$$

where $c_k \in (x_{k-1}, x_k)$. From this it is clear that

$$L\left(\sqrt{1 + f'(x)^2}, P\right) \leqslant l(P) \leqslant U\left(\sqrt{1 + f'(x)^2}, P\right). \tag{5}$$

Taking supreme on both sides of the first inequality we have

$$\int_a^b \sqrt{1 + f'(x)^2} \, \mathrm{d}x \leqslant l. \tag{6}$$

Exercise 2. Prove that $\sqrt{1 + f'(x)^2}$ is Riemann integrable on $[a, b]$.

On the other hand, we observe that

Exercise 3. Let P, Q be partitions of $[a, b]$. Then $l(P \cup Q) \geqslant l(P)$.

From this and the property of Riemann upper sum, together with (5), we have, for arbitrary partitions P, Q,

$$l(P) \leqslant l(P \cup Q) \leqslant U\big(\sqrt{1+f'(x)^2}, P \cup Q\big) \leqslant U\big(\sqrt{1+f'(x)^2}, Q\big). \tag{7}$$

As P, Q are arbitrary, we can take supreme on the left end and infimum on the right end, to conclude

$$l \leqslant \int_a^b \sqrt{1+f'(x)^2}\, \mathrm{d}x. \tag{8}$$

Thus the proof ends. □

- Arc length of parametrized curves.

  More generally, a curve is represented as

$$(x(t), y(t)), \qquad a \leqslant t \leqslant b. \tag{9}$$

If we similarly approximate by polygonal curves, we would finally reach

THEOREM 3. *Under the following assumptions on x, y,*

i. x, y are continuous on $[a, b]$;

ii. x', y' are continuous on (a, b);

iii. $\lim_{t \to a+} x'$, $\lim_{t \to a+} y'$, $\lim_{t \to b-} x'$ and $\lim_{t \to b-} y'$ exist and are finite,

there holds

$$l = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, \mathrm{d}t. \tag{10}$$

Exercise 4. Prove Theorem 3.

- Arc length of parametrized curves in polar coordinates.

  - Please first review polar coordinates. For example read the wiki page for "Polar coordinate system".

In this case we have $x(t) = r(t)\cos(\theta(t))$ and $y(t) = r(t)\sin(\theta(t))$ which leads to

THEOREM 4. *Under the following assumptions on r, θ,*

i. r, θ are continuous on $[a, b]$;

ii. r', θ' are continuous on (a, b);

iii. $\lim_{t \to a+} r'$, $\lim_{t \to a+} \theta'$, $\lim_{t \to b-} r'$ and $\lim_{t \to b-} \theta'$ exist and are finite,

there holds

$$l = \int_a^b \sqrt{r'(t)^2 + r(t)^2\, \theta'(t)^2}\, \mathrm{d}t. \tag{11}$$

In particular, when the curve is given by $r = r(\theta)$, $a \leqslant \theta \leqslant b$, the arc length is given by

$$l = \int_a^b \sqrt{r'(\theta)^2 + r(\theta)^2}\, \mathrm{d}\theta. \tag{12}$$

Exercise 5. Prove Theorem 4.

- Examples.

Example 5. Calculate the circumference of the unit circle $x^2+y^2=1$.

Solution.

- Method 1. We calculate the curve length l of the graph $y=\sqrt{1-x^2}, -1\leqslant x\leqslant 1$. Then the circumference is $2\,l$.

$$\begin{aligned} l &= \int_{-1}^{1}\sqrt{1+[(\sqrt{1-x^2})']^2}\,\mathrm{d}x \\ &= \int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x \\ &\overset{x=\sin t}{=\!=} \int_{-\pi/2}^{\pi/2}\mathrm{d}t=\pi. \end{aligned} \tag{13}$$

  So the circumference is $2\,\pi$.

  Exercise 6. Note that $f(x)=\sqrt{1-x^2}$ does not fully satisfy the hypotheses in Theorem 2. Explain why the above calculate is still reasonable and should give the correct answer.

- Method 2. We parametrize $x(t)=\cos t$, $y(t)=\sin t, 0\leqslant t<2\,\pi$. Then

$$l=\int_0^{2\pi}\sqrt{x'(t)^2+y'(t)^2}\,\mathrm{d}t=2\,\pi. \tag{14}$$

Exercise 7. Calculate the arc length of $x=\cos^3 t$, $y=\sin^3 t$, $t\in[0,2\,\pi)$.

Exercise 8. Calculate the arc length of the space curve

$$x=\cos t,\ y=\sin t,\ z=t. \tag{15}$$

Example 6. Calculate the arc length of $r=1+\cos\theta$, $0\leqslant\theta<2\,\pi$.

Solution. We calculate

$$\begin{aligned} l &= \int_0^{2\pi}\sqrt{r'(\theta)^2+r(\theta)^2}\,\mathrm{d}\theta \\ &= \int_0^{2\pi}\sqrt{2}\,\sqrt{1+\cos\theta}\,\mathrm{d}\theta \\ &= 2\int_0^{2\pi}\sqrt{\cos^2\frac{\theta}{2}}\,\mathrm{d}\theta \\ &= 2\left[\int_0^{\pi}\cos\frac{\theta}{2}\,\mathrm{d}\theta-\int_{\pi}^{2\pi}\cos\frac{\theta}{2}\,\mathrm{d}\theta\right] \\ &= 8. \end{aligned}$$