MATH 118 WINTER 2015 LECTURE 36 (MAR. 18, 2015)

- Recall
  - We will focus on
$$\min\ f(x) \qquad \text{subject to } a \leqslant x \leqslant b. \tag{1}$$
    Note. You should be able to "translate" everything into the context of $\max f(x)$ subject to $a \leqslant x \leqslant b$.
  - Global/local minimizer.
    - $x_0 \in [a, b]$ is a global minimizer: $\forall x \in [a, b],\ f(x) \geqslant f(x_0)$;
    - $x_0 \in [a, b]$ is a local minimizer: $\exists \delta > 0, \forall x \in [a, b] \cap (x_0 - \delta, x_0 + \delta),\ f(x) \geqslant f(x_0)$;
    - $x_0 \in [a, b]$ is an interior local minimizer:
      i. x_0 is a local minimizer;
      ii. $x_0 \in (a, b)$.

    Exercise 1. Prove that 0 is not a local minimizer of $\min x^3$ subject to $x \in \mathbb{R}$. (Sol:[1])
  - Necessary condition for x_0 to be a local minimizer.

    THEOREM. *Let x_0 be an interior local minimizer. Assume f is differentiable on (a, b). Then $f'(x_0) = 0$.*

    Exercise 2. Prove: $f'(x_0) = 0$ if x_0 is an interior local maximizer.

    Exercise 3. Find a $f(x)$ such that $f'(x_0) = 0$ for some x_0 but x_0 is neither a local minimizer nor a local maximizer.

    Example 1. Solve $\min x \ln^2 x$ subject to $0 \leqslant x < \infty$.

    Solution. Solving $(x \ln^2 x)' = 0$ we have $x_{1,2} = 1, e^{-1}$. Compare
$$f(0) = 0, \quad f(1) = 0, \quad f(e^{-1}) = e^{-1}, \quad f(\infty) = \infty \tag{2}$$
    we see that the global minimum is 0 with two global minimizers $0, 1$.

- Which solutions to $f'(x) = 0$ are local minimizers?
  - Note that when $f'(x_0) = 0$, there are three cases: x_0 is a local minimizer, a local maximizer, or neither.
  - First order conditions.

    THEOREM 2. *Let f be differentiable on (a, b). Let $f'(x_0) = 0$. Then:*
    - *If there is $\delta > 0$ such that $f'(x) \leqslant 0$ for $x \in (x_0 - \delta, x_0)$ and $f'(x) \geqslant 0$ for $x \in (x_0, x_0 + \delta)$, then x_0 is a local minimizer;*
    - *If there is $\delta > 0$ such that $f'(x) \geqslant 0$ for $x \in (x_0 - \delta, x_0)$ and $f'(x) \leqslant 0$ for $x \in (x_0, x_0 + \delta)$, then x_0 is a local maximizer;*

    Proof. We prove that first claim and leave the second one as exercise. Let $x \in (x_0 - \delta, x_0 + \delta)$ be arbitrary, we will prove $f(x) \geqslant f(x_0)$. There are three cases:
    1. $x = x_0$. We have $f(x) = f(x_0)$;

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1. Let $\delta > 0$ be arbitrary. Set $x = -\frac{\delta}{2}$. We have $f(x) = -\left(\frac{\delta}{2}\right)^3 < 0 = f(0)$.

2. $x\in(x_0-\delta,x_0)$. By MVT we have

$$\frac{f(x)-f(x_0)}{x-x_0}=f'(c) \tag{3}$$

for some $c\in(x,x_0)\subset(x_0-\delta,x_0)$. By assumption $f'(c)\leqslant 0$. As $x-x_0<0$ this implies $f(x)-f(x_0)\geqslant 0\Longrightarrow f(x)\geqslant f(x_0)$.

3. $x\in(x_0,x_0+\delta)$. The proof is almost identical to that of the previous case. □

Exercise 4. Let f be differentiable on (a,b). Let $f'(x_0)=0$. Further assume that there is $\delta>0$ such that $f'(x)>0$ for $x\in(x_0-\delta,x_0)$. Prove that x_0 is not a local minimizer.

Exercise 5. Let f be differentiable on (a,b). Assume that there is $\delta>0$ such that $f'(x)\leqslant 0$ for $x\in(x_0-\delta,x_0)$ and $f'(x)\geqslant 0$ for $x\in(x_0,x_0+\delta)$. Prove $f'(x_0)=0$.

Remark 3. Note that these conditions are not necessary. For example, $x=0$ is a local minimizer to

$$\min f(x)=x^2\left(1+\sin\frac{1}{x}\right), \qquad x\in\mathbb{R} \tag{4}$$

but for every $\delta>0$ there is $x\in(-\delta,0)$ such that $f'(x)>0$.

Example 4. We return to Example 1. We have $f'(x)=\ln x\,(\ln x+1)$. Thus

- At $x_1=1$: Take $\delta=\frac{1}{2}$. We have $f'(x)<0$ when $x\in\left(\frac{1}{2},1\right)$ and $f'(x)>0$ when $x\in\left(1,\frac{3}{2}\right)$. Thus 1 is a local minimizer.
- At $x_2=e^{-1}$. Take $\delta=e^{-1}$. We have $f'(x)>0$ when $x\in(0,e^{-1})$ and $f'(x)<0$ when $x\in(e^{-1},2\,e^{-1})$. Thus e^{-1} is a local maximizer.

○ Second order conditions.

THEOREM 5. *Let f be differentiable on (a,b). Let $f'(x_0)=0$. Further assume that $f''(x_0)$ exists. Then:*

- $f''(x_0)>0\Longrightarrow x_0$ *is a local minimizer* $\Longrightarrow f''(x_0)\geqslant 0$.
- $f''(x_0)<0\Longrightarrow x_0$ *is a local maximizer* $\Longrightarrow f''(x_0)\leqslant 0$.

Proof. Assume $f''(x_0)>0$. By definition this implies

$$\lim_{x\to x_0}\frac{f'(x)-f'(x_0)}{x-x_0}>0 \tag{5}$$

which implies there is $\delta>0$ such that for all $x\in(x_0-\delta,x_0+\delta)-\{x_0\}$, $\frac{f'(x)-f'(x_0)}{x-x_0}>0$. As $f'(x_0)=0$, we conclude that $f'(x)<0$ for $x\in(x_0-\delta,x_0)$ and $f'(x)>0$ for $x\in(x_0,x_0+\delta)$. Now the conclusion follows from Theorem 2. □

Exercise 6. Prove $f''(x_0)<0\Longrightarrow x_0$ is a local maximizer.

Exercise 7. Prove that $f''(x_0)>0\Longrightarrow\exists\delta>0,\forall x\in(x_0-\delta,x_0+\delta)-\{x_0\}, \quad f(x)>f(x_0)$. Then use this to prove x_0 is a local maximizer $\Longrightarrow f''(x_0)\leqslant 0$.

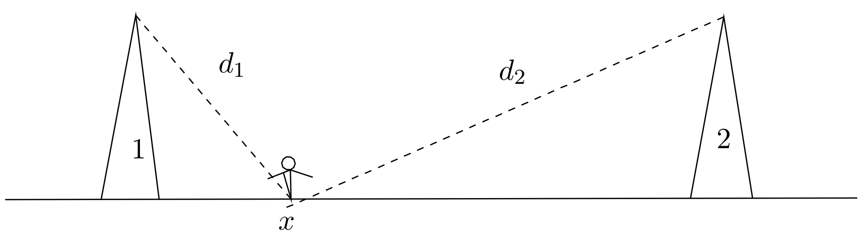
Example 6. We return to Example 1. We have $f''(x)=\frac{2\ln x+1}{x}$. Thus

- At $x_1=1$: $f''(1)=1>0$. Thus 1 is a local minimizer.
- At $x_2=e^{-1}$. $f''(e^{-1})=-e<0$. Thus e^{-1} is a local maximizer.

- Examples.

Example 7. (Wireless communication) Consider a user between two cell phone towers of height h with distance d part. Each tower broadcasts with power P. The user would like to receive the signal from tower 1 but not 2. Thus we would like to maximize the signal-to-noise ratio (SNR):

$$\text{SNR} = \frac{P/d_1^2}{P/d_2^2} = \frac{d_2^2}{d_1^2} = \frac{(d-x)^2+h^2}{x^2+h^2} \qquad \text{subject to } 0 \leqslant x \leqslant d. \tag{6}$$



Taking derivative we see that we should solve

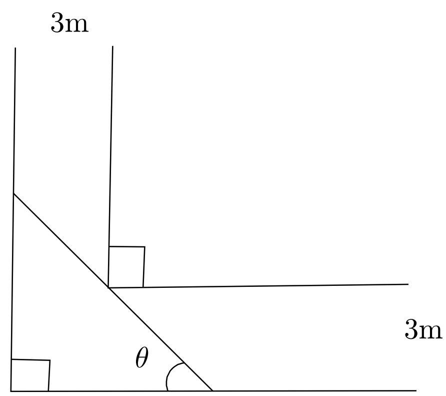
$$-2\,(d-x)\,(x^2+h^2) - 2\,x[(d-x)^2+h^2] = 0 \tag{7}$$

which gives $x_{1,2} = \frac{d \pm \sqrt{d^2+4\,h^2}}{2}$. Both are out of $[0, d]$. So we only need to check the end points $0, d$ and conclude that x should be 0.

Remark 8. This is quite silly. To make it (arguably) less silly we drop the constraint $0 \leqslant x \leqslant d$. This time we see that $x_2 := \frac{d - \sqrt{d^2+4\,h^2}}{2}$ is the global maximizer.

Exercise 8. Prove the above statement.

Example 9. (Carrying a pole in a hallway) We consider the following problem. We try to bring a long pole through the corner of the following hallway. What is the maximum length of the pole that allows us to do so?



It is easy to see that the optimization problem reads

$$\min l(\theta) = \frac{3}{\cos\theta} + \frac{3}{\sin\theta}, \qquad \text{subject to } 0 \leqslant \theta \leqslant \frac{\pi}{2}. \tag{8}$$

We calculate

$$l'(\theta) = \frac{3\sin\theta}{\cos^2\theta} - \frac{3\cos\theta}{\sin^2\theta} = \frac{3}{\cos^2\theta\sin^2\theta}[\sin^3\theta - \cos^3\theta]. \tag{9}$$

As

$$\sin^3\theta - \cos^3\theta = [\sin\theta - \cos\theta]\,[\sin^2\theta + \sin\theta\cos\theta + \cos^2\theta] = [\sin\theta - \cos\theta]\left[1 + \frac{1}{2}\sin(2\,\theta)\right] \tag{10}$$

the only solution to $l'(\theta) = 0$ is $\theta = \frac{\pi}{4}$. Thus the maximum length of the pole is $6\sqrt{2}$ meters.

Exercise 9. Try to study more complicated situations, for example the hallway could be 3D.