MATH 118 WINTER 2015 LECTURE 35 (MAR. 16, 2015)

- Mathematical optimization.
  - "Optimizing" a function $f(x)$ (called objective function) subject to certain restrictions on the variable x (called constraints).

    Example 1. Maximize the area of a triangle when the length of two sides are given a, b.

    Solution. Let the angle formed by these two sides be θ. Then we know that $A = \frac{1}{2} a\, b \sin\theta$ which is maximized when $\theta = \frac{\pi}{2}$.

  - We will focus on

$$\min/\max \ \ f(x) \qquad \text{subject to } a \leqslant x \leqslant b \quad (\text{or } a < x < b, a < x \leqslant b, a \leqslant x < b) \tag{1}$$

- Global and local minimum/minimizer (maximum/maximizer)
  - Global mininum/minimizer.

    DEFINITION 2. *$x_0 \in [a, b]$ is a global minimizer of (1) if and only if*

$$\forall x \in [a, b], \qquad f(x_0) \leqslant f(x). \tag{2}$$

In this case $f(x_0)$ is called the "global minimum" of the problem (1).

Exercise 1. Define global maximizer/maximum.

Exercise 2. Write down the working negation of "x is a global minimizer".

Example 3. Assuming that there is at least one global minimizer. Then there can be any number of global minimizers but exactly one global minimum.

For example,

$$f(x) = x^2 \qquad \text{subject to } -1 \leqslant x \leqslant 1 \tag{3}$$

has exactly one global minimizer, while

$$f(x) = \sin x \qquad \text{subject to } -\infty < x < \infty \tag{4}$$

has infinitely many global minimizers.

On the other hand, if there are two global minima $m_1 \neq m_2$, wlog $m_1 < m_2$. Let $f(x_1) = m_1$, $f(x_2) = m_2$. Then we have $x_1 \in [a, b]$ but $f(x_2) > f(x_1)$ which means x_2 is not a global minimizer and therefore m_2 is not a global minimum. Contradiction.

  - Local minimizer (maximizer).

    DEFINITION 4. *$x_0 \in [a, b]$ is a local minimizer of (1) if and only if*

$$\exists \delta > 0, \quad \forall x \in [a, b] \cap (x_0 - \delta, x_0 + \delta), \qquad f(x_0) \leqslant f(x). \tag{5}$$

Exercise 3. Define local maximizer.

Exercise 4. Write down the working negation of "x is a local minimizer".

  - Relation between global/local minimizers.

    LEMMA 5. *Let x_0 be a global minimizer of (1), then it is also a local minimizer of (1).*

Proof. Take $\delta = 1$. Let $x \in [a, b] \cap (x_0 - \delta, x_0 + \delta)$ be arbitrary. Then $x \in [a, b]$ and therefore x_0 being a global minimizer implies $f(x_0) \leqslant f(x)$. Thus ends the proof. □

Example 6. Consider

$$\min f(x) = -e^{-x^2} \cos x \qquad \text{subject to } -\infty < x < \infty. \tag{6}$$

Prove that there is a local minimizer that is not global.

Solution. First we prove that the global minimum is -1 and the only global minimizer is 0. Since

$$\forall x \in \mathbb{R}, \qquad -e^{-x^2} \cos x \geqslant -e^{-x^2} \geqslant -1 = f(0), \tag{7}$$

-1 is the global minimum and 0 is a global minimizer. To see that it is the only one, just notice that $x \neq 0 \Longrightarrow -e^{-x^2} > -1$.

Now we prove that there is a local minimizer in $\left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$. We see that $f\left(\frac{3\pi}{2}\right) = f\left(\frac{5\pi}{2}\right) = 0$ and $f(x) < 0$ on $\left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$. Since $f(x)$ is continuous on $\left[\frac{3\pi}{2}, \frac{5\pi}{2}\right]$ there is $x_0 \in \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$ such that $\forall x \in \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$, $f(x_0) \leqslant f(x)$. Now taking $\delta := \min\left\{\left|x_0 - \frac{3\pi}{2}\right|, \left|x_0 - \frac{5\pi}{2}\right|\right\}$ we see that $\forall x \in (x_0 - \delta, x_0 + \delta)$, $f(x_0) \leqslant f(x)$ which means x_0 is a local minimizer. Since 0 is the only global minimizer, x_0 is not a global minimizer.

Exercise 5. Prove that (6) has a local maximizer that is not global.

- Searching for global minimizer/maximizer.
  - Idea: Find all local minimizers and then compare them.
  - First order necessary condition.

DEFINITION 7. (INTERIOR LOCAL MINIMIZER) $x_0 \in [a, b]$ *is an interior local minimizer of (1) if and only if*

i. x_0 *is a local minimizer of (1);*

ii. $x_0 \in (a, b)$.

THEOREM 8. *Assume $f(x)$ is differentiable on (a, b). Let x_0 be an interior local minimizer of (1). Then $f'(x_0) = 0$.*

Proof. Let $\delta_1 := \min\{|x_0 - a|, |x_0 - b|\} > 0$. As x_0 is an interior local minimizer, there is $\delta_2 > 0$ such that $\forall x \in [a, b] \cap (x_0 - \delta_2, x_0 + \delta_2)$, $f(x_0) \leqslant f(x)$. Now set $\delta := \min\{\delta_1, \delta_2\}$, we have $\forall x \in (x_0 - \delta, x_0 + \delta)$, $f(x_0) \leqslant f(x)$.

Let $x \in (x_0 - \delta, x_0)$. We have $\frac{f(x) - f(x_0)}{x - x_0} \leqslant 0$. Taking limit $x \rightarrow x_0+$ we reach $f'(x_0) \leqslant 0$. Similar consideration for $x \in (x_0, x_0 + \delta)$ leads to $f'(x_0) \geqslant 0$. Therefore $f'(x_0) = 0$. □

Exercise 6. Prove that x_0 is an interior local maximizer of (1) $\Longrightarrow f'(x_0) = 0$.

Remark 9. $f'(x_0) = 0$ does not imply that x_0 is either a local minimizer or a local maximizer.

Exercise 7. Prove that 0 is neither a local minimizer nor a local maximizer for $f(x) = x^3$ over $\mathbb{R}$.

- Strategy: Solve $f'(x)=0$ to get all candidates for local minimizers, then compare them with $f(a)$, $f(b)$.

  Example 10. Solve $\min x^3-3x+3$ subject to $x\in\left[-3,\frac{3}{2}\right]$.

  Solution. Solving $(x^3-3x+3)'=0$ we have $x_{1,2}=\pm 1$. Now calculate

$$f(-3)=-15,\quad f\left(\frac{3}{2}\right)=\frac{15}{8},\quad f(1)=1,\quad f(-1)=5 \tag{8}$$

  therefore the global miminum is $f(-3)=-15$.

  Exercise 8. Solve $\max x^3-3x+3$ subject to $x\in\left[-3,\frac{3}{2}\right]$.