Math 118 Winter 2015 Midterm Exam 2 Solutions

Mar. 13, 2015 10am - 10:50am. Total 20+2 Pts

NAME: ID#:

- There are five questions.
- Please write clearly and show enough work.

Question 1. (5 pts) *Is* $\frac{\ln x}{x^2}$ *improperly integrable on* $(1, \infty)$*? Justify your claim.*

Solution.

First as $\frac{\ln x}{x^2}$ is continuous on $(1, \infty)$, it is locally integrable.

- By definition.

  We have

$$\int \frac{\ln x}{x^2} \mathrm{d}x = \int \ln x \, \mathrm{d}\left(-\frac{1}{x}\right) = -\frac{\ln x}{x} + \int \frac{\mathrm{d}x}{x^2} = -\frac{\ln x + 1}{x} + C. \tag{1}$$

  As

$$\lim_{d \to \infty} \int_1^d \frac{\ln x}{x^2} \mathrm{d}x = \lim_{d \to \infty} \left[-\frac{\ln d + 1}{d} + 1\right] = 1 \tag{2}$$

  we see that the function is improperly integrable on $(1, \infty)$.

- By comparison.

  We have

$$\left(2\, x^{1/2} - \ln x\right)' > 0 \tag{3}$$

  for $x \in (1, \infty)$, as $2 \cdot 1^{1/2} > \ln 1$, we have

$$0 < \ln x < 2\, x^{1/2} \tag{4}$$

  on $(0, \infty)$. Consequently

$$\left|\frac{\ln x}{x^2}\right| \leqslant 2\, x^{-3/2} \tag{5}$$

  and improper integrability follows.

Question 2. (5 pts) *Let* $f_n(x) := e^{-nx}\cos(n^2 x)$.

a) **(2 pts)** *Calculate* $\lim_{n\to\infty} f_n(x)$ *on* $(0,\infty)$*;*

b) **(3 pts)** *Is the convergence uniform? Justify your claim.*

Solution.

a) Let $x > 0$ be arbitrary. Then we have $0 \leqslant e^{-nx}\cos(n^2 x) \leqslant e^{-nx}$ and therefore $\lim_{n\to\infty} e^{-nx}\cos(n^2 x) = 0$ be Squeeze.

b) Set $x_n = \frac{2\pi}{n}$. Then we have

$$M_n := \sup_{x>0} |f_n(x) - 0| \geqslant |e^{-n x_n}\cos(n^2 x_n)| = e^{-2\pi}\cos(2n\pi) = e^{-2\pi}. \qquad (6)$$

Therefore the convergence is not uniform.

Question 3. (5 pts) *Let* $f(x) := \sum_{n=1}^{\infty} \frac{\sin(n^3 x)}{3^n}$.

a) **(1 pt)** *Prove that* $f(x)$ *is defined for all* $x \in \mathbb{R}$.

b) **(2 pts)** *Is* $f(x)$ *continuous on* $\mathbb{R}$*? Justify.*

c) **(2 pts)** *Is* $f(x)$ *differentiable on* $\mathbb{R}$*? Justify.*

Solution.

a) We have

$$\forall x \in \mathbb{R}, \qquad \left|\frac{\sin(n^3 x)}{3^n}\right| \leqslant \frac{1}{3^n}. \qquad (7)$$

As $\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges, $\sum_{n=1}^{\infty} \frac{\sin(n^3 x)}{3^n}$ converges uniformly on $\mathbb{R}$ and thus $f(x)$ is defined for all $x \in \mathbb{R}$.

b) As each $\frac{\sin(n^3 x)}{3^n}$ is continuous on $\mathbb{R}$ and $\sum_{n=1}^{\infty} \frac{\sin(n^3 x)}{3^n}$ converges uniformly on $\mathbb{R}$, $f(x)$ is continuous on $\mathbb{R}$.

c) We have

$$\left|\left(\frac{\sin(n^3 x)}{3^n}\right)'\right| = \left|\frac{n^3}{3^n}\cos(n^3 x)\right| \leqslant \frac{n^3}{3^n}. \qquad (8)$$

As $\lim_{n\to\infty} \frac{n^3}{(3/2)^n} = 0$ there is $M > 0$ such that $n^3 \leqslant M\left(\frac{3}{2}\right)^n$ for all $n \in \mathbb{N}$. Therefore

$$\left|\left(\frac{\sin(n^3 x)}{3^n}\right)'\right| \leqslant \frac{n^3}{3^n} \leqslant \frac{M}{2^n}. \qquad (9)$$

As $\sum_{n=1}^{\infty} \frac{M}{2^n}$ converges, $\sum_{n=1}^{\infty} \left(\frac{\sin(n^3 x)}{3^n}\right)'$ converges uniformly on $\mathbb{R}$. Therefore $f(x)$ is differentiable on $\mathbb{R}$.

Question 4. (5 pts) *Prove that* $\frac{e^{-x}-e^{-2x}}{x}$ *is improperly integrable on* $(0,\infty)$.

Solution. First as $\frac{e^{-x}-e^{-2x}}{x}$ is continuous on $(0,\infty)$, local integrability follows. Now by MVT we have

$$\forall x \in (0,\infty), \qquad \left|\frac{e^{-x}-e^{-2x}}{x}\right| = \frac{e^{-x}-e^{-2x}}{x} = e^{-c} \leqslant 1. \tag{10}$$

On the other hand we have

$$\forall x \in (0,\infty), \qquad \frac{e^{-x}-e^{-2x}}{x} \leqslant \frac{e^{-x}}{x}. \tag{11}$$

Therefore

$$\forall x \in (0,\infty), \qquad \left|\frac{e^{-x}-e^{-2x}}{x}\right| \leqslant \min\left\{1, \frac{e^{-x}}{x}\right\} \leqslant g(x) := \begin{cases} 1 & x \in (0,1) \\ e^{-x} & x \in [1,\infty) \end{cases} \tag{12}$$

As $g(x)$ is improperly integrable on $(0,\infty)$, $\frac{e^{-x}-e^{-2x}}{x}$ is improperly integrable on $(0,\infty)$ by comparison.

Question 5. (Extra 2 pts) *Calculate* $\int_0^{\infty} \frac{e^{-x}-e^{-2x}}{x}\,\mathrm{d}x$.

Solution. Let $0 < c < d < \infty$ be arbitrary. We have

$$\begin{aligned} \int_c^d \frac{e^{-x}-e^{-2x}}{x}\,\mathrm{d}x &= \int_c^d \frac{e^{-x}}{x}\,\mathrm{d}x - \int_c^d \frac{e^{-2x}}{x}\,\mathrm{d}x \\ &= \int_c^d \frac{e^{-x}}{x}\,\mathrm{d}x - \int_{2c}^{2d} \frac{e^{-x}}{x}\,\mathrm{d}x \\ &= \int_c^{2c} \frac{e^{-x}}{x}\,\mathrm{d}x - \int_d^{2d} \frac{e^{-x}}{x}\,\mathrm{d}x. \end{aligned} \tag{13}$$

Now for $d > 1$, we have

$$\int_d^{2d} \frac{e^{-x}}{x}\,\mathrm{d}x \leqslant \int_d^{2d} e^{-x}\,\mathrm{d}x = e^{-d} - e^{-2d} \longrightarrow 0 \text{ as } d \longrightarrow \infty. \tag{14}$$

On the other hand, we have, for $x > 0$, by MVT,

$$0 < \frac{1 - e^{-x}}{x} = e^{-\xi} < 1 \tag{15}$$

therefore

$$0 \leqslant \int_c^{2c} \frac{1 - e^{-x}}{x} \, \mathrm{d}x \leqslant \int_c^{2c} \mathrm{d}x = c \longrightarrow 0 \text{ as } c \longrightarrow 0+. \tag{16}$$

As $\int_c^{2c} \frac{1}{x} \, \mathrm{d}x = \ln 2$ for all $c > 0$, there holds $\int_c^{2c} \frac{e^{-x}}{x} \, \mathrm{d}x \longrightarrow \ln 2$ as $c \longrightarrow 0+$ and consequently

$$\int_0^\infty \frac{e^{-x} - e^{-2x}}{x} \, \mathrm{d}x = \ln 2. \tag{17}$$