Math 118 Winter 2015 Midterm 2 Review

- Midterm 1 coverage:
  - Lectures 20 - 31 and the exercises therein.
  - Homeworks 5 – 7.
  - The exercises below are to help you on the concepts and techniques. The exam problems may or may not look like these exercises/problems.

1. Exercises.

Exercise 1. Prove/disprove **by definition** the convergence of the following improper integrals.

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^2}; \qquad \int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x}}; \qquad \int_0^1 \frac{\mathrm{d}x}{(x^2-1)^2}; \tag{1}$$

Exercise 2. Prove/disprove the convergence of the following improper integrals.

$$\int_1^{\infty} \frac{\mathrm{d}x}{x^2\,(1+e^x)}; \qquad \int_1^{\infty} \frac{\sin x}{x^3}\,\mathrm{d}x; \qquad \int_0^1 \frac{\mathrm{d}x}{\sqrt{x}+x^3}; \tag{2}$$

Exercise 3. Let $f(x)$, $g(x)$ be improperly integrable on $(0,\infty)$. Prove **by definition**: If $f(x) \leqslant g(x)$ for all $x \in (0,\infty)$, then $\int_0^{\infty} f(x)\,\mathrm{d}x \leqslant \int_0^{\infty} g(x)\,\mathrm{d}x$.

Exercise 4. Calculate the following limits of functions for $x \in \mathbb{R}$. Then determine whether the convergence is uniform or not. Justify.

$$\lim_{n\to\infty} n\,x^2\,e^{-n^2 x}; \qquad \lim_{n\to\infty} \frac{\sin(\sqrt{n}\,x)}{\ln n}; \qquad \lim_{n\to\infty} \frac{1-\cos\left(\frac{x}{n}\right)}{1+(\sin x)^2}. \tag{3}$$

Exercise 5. For each of the following series, find all $x \in \mathbb{R}$ such that the series converges. Then discuss the continuity of the functions defined by these series.

$$\sum_{n=1}^{\infty} \frac{n-1}{n+1}\left(\frac{x}{3\,x+1}\right)^n; \qquad \sum_{n=1}^{\infty} n\,e^{-nx}; \qquad \sum_{n=1}^{\infty} \frac{x^n}{1+x^{2n}} \tag{4}$$

Exercise 6. Find bounded functions $f_n(x)\colon \mathbb{R} \mapsto \mathbb{R}$ such that

$$\lim_{n\to\infty} f_n(x) = f(x) \tag{5}$$

for every $x \in \mathbb{R}$ but $f(x)$ is not bounded.

2. More exercises.

Exercise 7. Prove/disprove the convergence of $\int_0^{\infty} \frac{\sin x}{x^2}\,\mathrm{d}x$ and $\int_0^{\infty} \frac{\mathrm{d}x}{x^2\,(1+e^x)}$.

Exercise 8. Let $f(x)$ be such that $\lim_{d\longrightarrow+\infty} \int_{-d}^{d} f(x)\,\mathrm{d}x$ exists and is finite. Does it follow that $f(x)$ is improperly integrable on $(-\infty,\infty)$? Justify your claim.

Exercise 9. Prove or disprove: If $f(x)$ is improperly integrable on $(0,\infty)$, then $\lim_{x\to\infty} f(x) = 0$.

Exercise 10. Find all $p \in \mathbb{R}$ such that $\sum_{n=2}^{\infty} \frac{1}{n\,(\ln n)^p}$ converges. Justify.

Exercise 11. Find all $x \in \mathbb{R}$ such that $\sum_{n=1}^{\infty} \frac{(n+x)^n}{n^{n+x}}$ is convergent and study the continuity of the function defined by this series.

Exercise 12. Prove:

a) $\sum_{n=1}^{\infty} (-1)^n\,x^n\,(1-x)$ converges uniformly on $[0,1]$;

b) $\sum_{n=1}^{\infty} |(-1)^n x^n (1-x)|$ converges on $[0,1]$ but the convergence is not uniform.

Exercise 13. Let $f_n(x) = x \arctan(n x)$.

a) Prove that $\lim_{n\to\infty} f_n(x) = \frac{\pi}{2} |x|$;

b) Prove that $\lim_{n\to\infty} f_n'(x)$ exists for every x, including $x = 0$, but the convergence is not uniform in any interval containing 0.

Exercise 14. Let $f_n(x)$ be continuous on $[a, b]$ and assume $f_n \longrightarrow f$ uniformly on (a, b). Prove that $f_n \longrightarrow f$ uniformly on $[a, b]$.

Exercise 15. Find the elementary functions given by

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}; \qquad \sum_{n=0}^{\infty} (-1)^{n-1} \frac{x^{2n+1}}{2n+1}; \qquad \sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}. \tag{6}$$

Justify your calculation.

3. Problems.

Problem 1. Find $f(x)$ that is improperly integrable on $(0,1)$ but $|f(x)|$ is not.

Problem 2. Find $f(x), g(x) > 0$ such that

a) $f(x)$ is improperly integrable on $(0,\infty)$ but $f(x)^2$ is not.

b) $g(x)^2$ is improperly integrable on $(0,\infty)$ but $g(x)$ is not.

Justify.

Problem 3. Let $S_0(x) = 1$ and define successively

$$S_n(x) = \sqrt{x\, S_{n-1}(x)}. \tag{7}$$

Prove that $S_n(x)$ converges uniformly on $[0,1]$.

Problem 4. Let $f_n(x) = g(x)\, x^n$ where $g(x)$ is continuous on $[0,1]$ with $g(1) = 0$. Prove that $f_n(x) \longrightarrow 0$ uniformly on $[0,1]$.

Problem 5. Show that there is no f satisfying both of the following:

- There is a sequence of numbers $\{a_n\}$ such that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ on $(-1,1)$;
- $f\left(\frac{1}{n}\right) = \frac{\sin n}{n^2}$ for all $n \in \mathbb{N}$.