MATH 118 WINTER 2015 LECTURE 34 (MAR. 12, 2015)

Midterm 2 Review III

- Checking uniform convergence for infinite series of functions.
  - Check that $S_n(x) := \sum_{k=1}^n u_k(x)$ (as a sequence) converges uniformly on $[a, b]$;
  - Weierstrass's M-test.

    If $|u_n(x)| \leqslant a_n$ on $[a,\ b]$ and $\sum_{n=1}^\infty a_n$ converges, then $\sum_{n=1}^\infty u_n(x)$ converges uniformly on $[a, b]$.

    Exercise 1. $\sum_{n=1}^\infty u_n(x)$ converges uniformly on $[a, b]$ if $\sum_{n=1}^\infty \sup_{[a,b]} |u_n(x)|$ converges.

    Exercise 2. Find a uniformly convergent $\sum_{n=1}^\infty u_n(x)$ such that Weierstrass's M-test fails.
  - A necessary condition.

    If $\sum_{n=1}^\infty u_n(x)$ converges uniformly on $[a, b]$, then $u_n(x) \longrightarrow 0$ uniformly on $[a, b]$.

Example 1. Study the convergence of $\sum_{n=1}^\infty \frac{x^n}{1+x^{2n}}$ on $(0, \infty)$.

Solution.

- Convergence. Three cases.
  - $x \in (0, 1)$. In this case we have $\left|\frac{x^n}{1+x^{2n}}\right| \leqslant x^n$ and convergence follows.
  - $x = 1$. In this case $\frac{x^n}{1+x^{2n}} = \frac{1}{2}$ and $\sum_{n=1}^\infty \frac{x^n}{1+x^{2n}}$ diverges.
  - $x > 1$. In this case we have $\left|\frac{x^n}{1+x^{2n}}\right| < \left(\frac{1}{x}\right)^n$. As $\left|\frac{1}{x}\right| < 1$ convergence follows.
- Uniform convergence.
  - Convergence is not uniform on $(0,\ 1)$. We calculate $\sup_{x\in(0,1)} \left|\frac{x^n}{1+x^{2n}}\right| \geqslant \lim_{x\to 1-} \frac{x^n}{1+x^{2n}} = \frac{1}{2}$. Therefore $\frac{x^n}{1+x^{2n}}$ does not converge to 0 uniformly on $(0, 1)$ and the convergence of $\sum_{n=1}^\infty \frac{x^n}{1+x^{2n}}$ on $(0, 1)$ is thus not uniform.
  - Convergence is not uniform on $(1, \infty)$. The proof is similar to the $(0, 1)$ case.

Exercise 3. Let $0 < a < 1$. Prove that $\sum_{n=1}^\infty \frac{x^n}{1+x^{2n}}$ converges uniformly on $(0, a)$.

- Properties of uniformly convergent sequence/series of functions.
  - Continuity.

    Let $f(x) := \lim_{n\to\infty} f_n(x)$ (or $\sum_{n=1}^\infty u_n(x)$) on $[a, b]$. If

    i. each $f_n(x)$ ($u_n(x)$) is continuous on $[a, b]$, and

    ii. the convergence is uniform on $[a, b]$,

    then $f(x)$ is continuous on $[a, b]$.
  - Integrability.

    Let $f(x) := \lim_{n\to\infty} f_n(x)$ (or $\sum_{n=1}^\infty u_n(x)$) on $[a, b]$. If

    i. each $f_n(x)$ ($u_n(x)$) is Riemann integrable on $[a, b]$, and

    ii. the convergence is uniform on $[a, b]$,

    then $f(x)$ is Riemann integrable on $[a, b]$.

- Differentiability.

  Let $f(x) := \lim_{n\to\infty} f_n(x)$ (or $\sum_{n=1}^{\infty} u_n(x)$) and $\varphi(x) := \lim_{n\to\infty} f_n'(x)$ ($\sum_{n=1}^{\infty} u_n'(x)$) on $[a,b]$. If

  - both convergences to f and φ are uniform on $[a,b]$,

  then $f'(x) = \varphi(x)$ on $[a,b]$.

Example 2. Study $f(x) := \sum_{n=1}^{\infty} 2^n \sin\left(\frac{1}{3^n x}\right)$ on $(0,\infty)$.

Solution.

- Domain.

  We have $\left|2^n \sin\left(\frac{1}{3^n x}\right)\right| \leqslant \frac{2^n}{3^n x} = \frac{1}{x}\left(\frac{2}{3}\right)^n$ therefore $f(x)$ is defined for all $x \in (0,\infty)$.

- Uniform convergence.

  We have $M_n := \sup_{(0,\infty)} \left|2^n \sin\left(\frac{1}{3^n x}\right)\right| \geqslant 2^n \sin\left(\frac{1}{3^n 3^{-n}}\right) = 2^n \sin 1$. Thus $2^n \sin\left(\frac{1}{3^n x}\right)$ does not converge to 0 uniformly on $(0, \infty)$ and therefore $\sum_{n=1}^{\infty} 2^n \sin\left(\frac{1}{3^n x}\right)$ does not converge uniformly on $(0,\infty)$.

- From $\left|2^n \sin\left(\frac{1}{3^n x}\right)\right| \leqslant \frac{1}{x}\left(\frac{2}{3}\right)^n$ it is clear that the convergence is uniform on (a,∞) for every $a > 0$ and consequently $f(x)$ is continuous on $(0,\infty)$.

Exercise 4. Prove that $f(x)$ is differentiable on $(0,\infty)$.

Problem 1. Is $f(x)$ infinitely differentiable or improperly integrable on $(0,\infty)$? Justify.

Problem 2. Is $f(x) := \sum_{n=1}^{\infty} \frac{\sin(n^2 x)}{2^n}$ improperly integrable on $(0,\infty)$? Justify.

Example 3. Prove or disprove: Let $f(x) := \lim_{n\to\infty} f_n(x)$ (or $\sum_{n=1}^{\infty} u_n(x)$) on (a,b). If

i. each $f_n(x)$ ($u_n(x)$) is improperly integrable on (a,b), and

ii. the convergence is uniform on (a,b),

then $f(x)$ is improperly integrable on (a,b).

Solution. The claim is false. A counterexample is $f(x) = \frac{1}{x}$ and $f_n(x) = \frac{e^{-x/n}}{x}$ on $(1,\infty)$.

- Power series.

  Let $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ be a power series with radius of convergence $R := \left[\limsup_{n\to\infty} |a_n|^{1/n}\right]^{-1}$. Define $f(x) := \sum_{n=0}^{\infty} a_n (x - x_0)^n$. Then

  a) $f(x)$ is continuous on $(x_0 - R, x_0 + R)$.

  b) $f(x)$ is differentiable on $(x_0 - R, x_0 + R)$, and $f'(x) = \sum_{n=0}^{\infty} (n+1)\, a_{n+1} (x - x_0)^n$ on $(x_0 - R, x_0 + R)$.

  c) $f(x)$ is infinitely differentiable on $(x_0 - R, x_0 + R)$, and

  $$f^{(m)}(x) = \sum_{n=0}^{\infty} (n+m) \cdots (n+1)\, a_{n+m} (x - x_0)^n. \tag{1}$$

  d) Let $[a,b] \subset (x_0 - R, x_0 + R)$ be arbitrary. Then $f(x)$ is Riemann integrable on $[a,b]$ and furthermore

  $$\int_a^b f(x)\,\mathrm{d}x = \sum_{n=0}^{\infty} a_n \int_a^b (x - x_0)^n\,\mathrm{d}x. \tag{2}$$