MATH 118 WINTER 2015 LECTURE 33 (MAR. 11, 2015)

Midterm 2 Review: Infinite series of functions

- Definitions.
  - For sequences:
    - A number sequence $\{a_n\}$ convergens to $a \in \mathbb{R}$:
$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N}, \quad \forall n > N, \qquad |a_n - a| < \varepsilon. \tag{1}$$
    - A function sequence $\{f_n(x)\}$ converges to $f(x)$ on $[a, b]$:
$$\forall x \in [a, b], \qquad \lim_{n \to \infty} f_n(x) = f(x). \tag{2}$$
Equivalently,
$$\forall x \in [a, b], \quad \forall \varepsilon > 0, \quad \exists N \in \mathbb{N}, \quad \forall n > N, \qquad |f_n(x) - f(x)| < \varepsilon. \tag{3}$$
    - A function sequence $\{f_n(x)\}$ converges uniformly to $f(x)$ on $[a, b]$:
$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N}, \quad \forall n > N, \quad \forall x \in [a, b], \qquad |f_n(x) - f(x)| < \varepsilon. \tag{4}$$
  - For series:
    - A series $\sum_{n=1}^{\infty} a_n$ converges to $s \in \mathbb{R}$: Let $s_n := a_1 + a_2 + \cdots + a_n$, $\lim_{n \to \infty} s_n = s$.
    - A function series $\sum_{n=1}^{\infty} u_n(x)$ converges to $f(x)$ on $[a, b]$:
$$\forall x \in [a, b], \qquad \sum_{n=1}^{\infty} u_n(x) = f(x) \tag{5}$$
or equivalently,
$$\forall x \in [a, b], \qquad \lim_{n \to \infty} S_n(x) = f(x) \tag{6}$$
where $S_n(x) := u_1(x) + u_2(x) + \cdots + u_n(x)$.
    - A function series $\sum_{n=1}^{\infty} u_n(x)$ converges to $f(x)$ uniformly on $[a, b]$: $S_n(x)$ converges uniformly to $f(x)$ on $[a, b]$.

Example 1. Find all $x \in \mathbb{R}$ such that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(\frac{1-x}{1+x}\right)^n$ converges.

Solution. Let $x \in \mathbb{R}$ be arbitrary. Set $r := \left(\frac{1-x}{1+x}\right)^n$. We know that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} r^n$ converges for $|r| < 1$ and diverges for $|r| > 1$. At $r = 1$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which is convergent, while at $r = -1$ we have $\sum_{n=1}^{\infty} \frac{1}{n}$ which is divergent. Therefore $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(\frac{1-x}{1+x}\right)^n$ converges if and only if $-1 < \frac{1-x}{1+x} \leqslant 1$ which is equivalent to $x \geqslant 0$.

- Checking uniform convergence of a sequence of functions.
  - Methods.
    1. By definition:
       First calculate $f(x) = \lim_{n \to \infty} f_n(x)$, then study whether it is true that
$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N}, \quad \forall n > N, \quad \forall x \in [a, b], \qquad |f_n(x) - f(x)| < \varepsilon. \tag{7}$$
    2. By Cauchy:

Study whether it is true that

$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N}, \quad \forall m, n > N, \quad \forall x \in [a, b], \qquad |f_m(x) - f_n(x)| < \varepsilon. \tag{8}$$

3. A practical method:
   - First calculate $f(x) = \lim_{n \to \infty} f_n(x)$;
   - Then calculate $M_n := \sup_{x \in [a,b]} |f_n(x) - f(x)|$;
   - If $\lim_{n \to \infty} M_n = 0$ then $f_n(x) \longrightarrow f(x)$ uniformly on $[a, b]$, otherwise the convergence is not uniform.

Example 2. Let $f_n(x) = \frac{n\,x}{1 + n + x}$.

a) Calculate $\lim_{n \to \infty} f_n(x)$ on $(0, \infty)$.

b) Is the convergence uniform? Justify your claim.

Solution.

a) Let $x \in (0, \infty)$ be arbitrary. We have

$$\lim_{n \to \infty} \frac{n\,x}{1 + n + x} = \lim_{n \to \infty} \left(\frac{n}{n + (1 + x)} \right) x = x. \tag{9}$$

b) We have

$$\left| \frac{n\,x}{1 + n + x} - x \right| = \frac{x + x^2}{1 + n + x}. \tag{10}$$

Thus

$$M_n \geqslant \frac{n + n^2}{1 + n + n} > \frac{n}{3\,n} = \frac{1}{3}. \tag{11}$$

Thus $\lim_{n \to \infty} M_n$, if exists, must be greater or equal to $\frac{1}{3}$. Therefore $\lim_{n \to \infty} M_n = 0$ does not hold and the convergence is not uniform.

Exercise 1. Let $R > 0$ be arbitrary. Does $\frac{n\,x}{1 + n + x}$ converge to x uniformly on $(0, R)$?