Math 118 Winter 2015 Lecture 32 (Mar. 9, 2015)

Midterm 2 Review: Improper Integration

- Improper integrability.
  - A function $f(x)$ is improperly integrable on (a, b) if and only if
    i. $f(x)$ is locally integrable on (a, b), that is $f(x)$ is Riemann integrable on every $[c, d] \subset (a, b)$;
    ii. The double limit
    $$\lim_{d \longrightarrow b-} \left[\lim_{c \longrightarrow a+} \int_c^d f(x)\, \mathrm{d}x \right] \tag{1}$$
    exists and is finite.

    Exercise 1. Let $a, b \in \mathbb{R}$ and $f(x)$ be bounded on (a, b). Then f is improperly integrable on (a, b) if and only if f is Riemann integrable on (a, b).
  - Equivalent condition for ii.
    There is $x_0 \in (a, b)$ such that both limits
    $$\lim_{c \to a+} \int_c^{x_0} f(x)\, \mathrm{d}x \text{ and } \lim_{d \longrightarrow b-} \int_{x_0}^d f(x)\, \mathrm{d}x \tag{2}$$
    exist and are finite.
- Checking improper integrability.
  - Checking i.
    - If $f(x)$ is continuous on (a, b) then i is satisfied;
    - If $f(x)$ is monotone on (a, b) then i is satisfied.

    If $f(x)$ is neither monotone nor continuous, then more work needs to be done to check the local integrability.
  - Methods for checking ii.
    i. By definition. To do this, we need to first calcualte the indefinite integral $\int f(x)\, \mathrm{d}x = F(x) + C$ and then study $\lim_{d \to b-} [\lim_{c \to a+} (F(d) - F(c))]$.
    ii. By Cauchy. For example, if $f(x)$ is Riemann integrable on $[a, d]$ for every $d \in (a, b)$, then f is improperly integrable on (a, b) if and only if
    $$\forall \varepsilon > 0, \exists d_0 \in (a, b), \forall d_1, d_2 \in (d_0, b), \qquad \left| \int_{d_1}^{d_2} f(x)\, \mathrm{d}x \right| < \varepsilon. \tag{3}$$
    iii. By comparison.
      - If $|f(x)| \leqslant g(x)$ and $g(x)$ is improperly integrable on (a, b), so is f.
      - If $f(x) \geqslant |g(x)|$ and $g(x)$ is not improperly integrable on (a, b), then f is not improperly integrable on (a, b).

      Exercise 2. Prove the second claim.
      - In practice, usually compare with x^α for some $\alpha \in \mathbb{R}$.

      Exercise 3. Let $f(x): (0, \infty) \mapsto \mathbb{R}$ be locally integrable and such that
      $$|f(x)| \leqslant c_1\, x^{\alpha_1} \text{ on } (0, 1); \qquad |f(x)| \leqslant c_2\, x^{\alpha_2} \text{ on } (1, \infty) \tag{4}$$

for some $c_1, c_2 > 0$, $\alpha_1 > -1$ and $\alpha_2 < -1$, then $f(x)$ is improperly integrable on $(0, \infty)$.

Exercise 4. Let $f(x): (0, \infty) \mapsto \mathbb{R}$ be locally integrable and such that

$$\lim_{x \to 0+} \frac{|f(x)|}{x^{\alpha_1}} = c_1, \quad \lim_{x \to \infty} \frac{|f(x)|}{x^{\alpha_2}} = c_2 \tag{5}$$

for some $c_1, c_2 > 0$, $\alpha_1 > -1$ and $\alpha_2 < -1$, then $f(x)$ is improperly integrable on $(0, \infty)$.

iv. Dirichlet and Abel.

Note. We will only state the rough idea here. Please check Lecture 23 (FEb. 13, 2015) for exact statements of these theorems.

- Dirichlet. $\int_{d_1}^{d_2} f(x)\,\mathrm{d}x$ uniformly bounded in d_1, d_2, $g(x)$ monotone and $\lim_{x \to \infty} g(x) = 0$, then $f g$ is improperly integrable on $(0, \infty)$.
- Abel. $f(x)$ is improperly integrable on $(0, \infty)$, $g(x)$ monotone and bounded, then $f g$ is improperly integrable on $(0, \infty)$.

Exercise 5. Prove that $\frac{\sin x}{x^\alpha}$ is improperly integrable on $(1, \infty)$ when $\alpha > 0$.

- Examples.

Example 1. Is $\frac{\ln x}{(1 + x^2)}$ improperly integrable on $(0, \infty)$?

Solution. Yes. We have

$$\left| \frac{\ln x}{(1 + x)^2} \right| \leqslant |\ln x| \leqslant c_1 x^{-1/2} \text{ on } (0, 1) \tag{6}$$

and

$$\left| \frac{\ln x}{(1 + x)^2} \right| \leqslant \frac{|\ln x|}{x^2} \leqslant c_2 x^{-3/2} \text{ on } (1, \infty) \tag{7}$$

therefore the function is improperly integrable on $(0, \infty)$.

Example 2. Is $\sqrt{\tan x}$ improperly integrable on $\left(0, \frac{\pi}{2}\right)$?

Solution. Yes. We make a change of variable: $t = \tan x$. Then

$$\int_0^{\pi/2} \sqrt{\tan x}\,\mathrm{d}x = \int_0^\infty \frac{\sqrt{t}}{1 + t^2}\,\mathrm{d}t. \tag{8}$$

As

$$\left| \frac{\sqrt{t}}{1 + t^2} \right| \leqslant \frac{\sqrt{t}}{t^2} = t^{-3/2} \text{ on } (1, \infty) \tag{9}$$

and

$$\left| \frac{\sqrt{t}}{1 + t^2} \right| \leqslant \sqrt{t} \leqslant 1 \text{ on } (0, 1) \tag{10}$$

the improper integrability follows.

Example 3. Is $\frac{1 - \cos x}{x^2}$ improperly integrable on $(0, \infty)$?

Solution. Yes. We have on $(1, \infty)$ $\left| \frac{1 - \cos x}{x^2} \right| \leqslant 2\,x^{-2}$ and on $(0, 1)$, by Taylor expansion,

$$\left| \frac{1 - \cos x}{x^2} \right| = \left| \frac{\frac{\cos c}{2!} x^2}{x^2} \right| \leqslant \frac{1}{2}. \tag{11}$$

Exercise 6. Is $\ln\left(1 + \frac{1}{x}\right) - \frac{1}{1 + x}$ improperly integrable on $(1, \infty)$?