MATH 118 WINTER 2015 LECTURE 31 (MAR. 6, 2015)

- (VAN DER WAERDEN'S EXAMPLE) Define $u_0(x)$ through:

$$u_0(x) = |x| \text{ on } \left[-\frac{1}{2}, \frac{1}{2}\right], \qquad \forall x \in \mathbb{R}, u_0(x) = u_0(x+1). \tag{1}$$

Exercise 1. Prove that $u_0(x) = \min_{n \in \mathbb{Z}} |x - n|$.

Now define

$$f(x) := \sum_{n=1}^{\infty} 4^{-n} u_0(4^n x). \tag{2}$$

Exercise 2. Prove that $f(x)$ is continuous on $\mathbb{R}$.

We prove that $f(x)$ is nowhere differentiable. Take any $x \in [0, 1)$. For each $m \in \mathbb{N}$ we define $h_m = \epsilon_m 4^{-m}$ and determine the sign $\epsilon_m = \pm 1$ as follows. Divide $[0,1)$ into 4^m intervals $[0, 4^{-m})$, $[4^{-m}, 2 \cdot 4^{-m})$, ... Let x be in the k-th interval. If k is odd, we set $\epsilon_m = 1$, if k is even, we set $\epsilon_m = -1$.

Now we easily see that

$$f(x + h_m) - f(x) = \sum_{n=1}^{m-1} 4^{-n} [u_0(4^n x + 4^n h_m) - u_0(4^n x)]. \tag{3}$$

Furthermore by our construction we have

$$\frac{4^{-n} [u_0(4^n x + 4^n h_m) - u_0(4^n x)]}{h_m} = \pm 1 \tag{4}$$

for all m, n. Consequently

$$\frac{f(x + h_m) - f(x)}{h_m} = \sum_{n=1}^{m-1} \frac{4^{-n} [u_0(4^n x + 4^n h_m) - u_0(4^n x)]}{h_m} \tag{5}$$

is odd when m is even and even when m is odd. Therefore $\lim_{m \to \infty} \frac{f(x + h_m) - f(x)}{h_m}$ does not exist.

Exercise 3. Prove that if $\frac{f(x + h_m) - f(x)}{h_m}$ is odd when m is even and even when m is odd then $\lim_{m \to \infty} \frac{f(x + h_m) - f(x)}{h_m}$ does not exist.

Problem 1. Let

$$g(x) := \sum_{n=1}^{\infty} 10^{-n} u_0(10^n x). \tag{6}$$

Prove that $g(x)$ is continuous on $\mathbb{R}$ but nowhere differentiable. Does

$$h(x) := \sum_{n=1}^{\infty} 5^{-n} u_0(5^n x) \tag{7}$$

have the same property?

- (WEIERSTRASS'S EXAMPLE) van der Waerden's construction above is in fact a simplified version of the construction by Karl Weierstrass (1815 - 1897) in 1872, which is the first such "everywhere continuous nowhere differentiable" function ever constructed and shocked the whole mathematical community.

Weierstrass' original example is

$$f(x) := \sum_{n=1}^{\infty} b^n \cos(a^n \pi x) \tag{8}$$

where $b \in (0,1)$ and a is an odd integer with $a\,b > 1 + \frac{3\,\pi}{2}$.

Example 1. $f(x) := \sum_{n=1}^{\infty} \frac{\cos(21^n\,\pi\,x)}{3^n}$ is continuous on $\mathbb{R}$ but nowhere differentiable.

Proof. (METHOD 1) Continuity follows easily from the uniform convergence of the series.

Exercise 4. Prove that $f(x)$ is continuous on $\mathbb{R}$.

Now fix $r \in \mathbb{R}$ we will prove $f(x)$ is not differentiable at r.
For every $m \in \mathbb{N}$, let $\alpha_m \in \mathbb{Z}$ be such that

$$\alpha_m - \frac{1}{2} < 21^m\,r \leqslant \alpha_m + \frac{1}{2}. \tag{9}$$

Set

$$\varepsilon_m := 21^m\,r - \alpha_m \in \left(-\frac{1}{2}, \frac{1}{2}\right], \qquad h_m := \frac{1 - \varepsilon_m}{21^m} \in \left(\frac{1/2}{21^m}, \frac{3/2}{21^m}\right). \tag{10}$$

Now consider

$$\begin{aligned}\frac{f(r + h_m) - f(r)}{h_m} &= \sum_{k=1}^{m-1} \frac{\cos(21^k\,\pi\,r + 21^k\,\pi\,h_m) - \cos(21^k\,\pi\,r)}{3^k\,h_m} \\ &\quad + \sum_{k=m}^{\infty} \frac{\cos(21^k\,\pi\,r + 21^k\,\pi\,h_m) - \cos(21^k\,\pi\,r)}{3^k\,h_m} \\ &=: A + B.\end{aligned} \tag{11}$$

Exercise 5. Prove that $|A| \leqslant \frac{\pi}{6} \cdot 7^m$. (Hint:[1]).

For B, notice that for $k \geqslant m$,

$$\cos(21^k\,\pi\,r + 21^k\,\pi\,h_m) = \cos(21^{k-m}\,(\alpha_m + 1)\,\pi) = (-1)^{\alpha_m + 1}; \tag{12}$$

$$\cos(21^k\,\pi\,r) = \cos(21^{k-m}\pi\,\alpha_m + 21^{k-m}\,\pi\varepsilon_m) = (-1)^{\alpha_m} \cos(21^{k-m}\,\pi\,\varepsilon_m). \tag{13}$$

Therefore (recalling (10): $|\varepsilon_m| \leqslant \frac{1}{2}$)

$$|B| = \frac{1}{h_m} \left| \sum_{k=m}^{\infty} \frac{1 + \cos(21^{k-m}\,\pi\,\varepsilon_m)}{3^k} \right| > \frac{1}{h_m} \frac{1 + \cos(\pi\,\varepsilon_m)}{3^m} \geqslant \frac{1}{h_m\,3^m} > \frac{2}{3}\,7^m. \tag{14}$$

As $\frac{2}{3} > \frac{\pi}{6}$ we see that

$$\left| \frac{f(r + h_m) - f(r)}{h_m} \right| > \left(\frac{2}{3} - \frac{\pi}{6} \right) 7^m \longrightarrow \infty \text{ as } m \longrightarrow \infty. \tag{15}$$

Consequently $\lim_{h \to 0} \frac{f(r+h) - f(r)}{h}$ cannot exist and $f(x)$ is not differentiable at r. □

Proof. (METHOD 2) We take $h_m := \frac{2\,s}{21^{m+1}}$ where $s < \frac{63}{4}$ is a natural number. Observe that when $k \geqslant m + 1$, we have

$$21^k\,h_m = 2\,s\,21^{k-m-1} \tag{16}$$

is even and therefore

$$\cos(21^k\,\pi\,(r + h_m)) = \cos(21^k\,\pi\,r). \tag{17}$$

1. MVT.

This gives

$$\begin{aligned}\frac{f(r+h_m)-f(r)}{h_m} &= \sum_{k=1}^{m} \frac{\cos(21^k\pi r+21^k\pi h_m)-\cos(21^k\pi r)}{3^k h_m}\\ &= \sum_{k=1}^{m-1} \frac{\cos(21^k\pi r+21^k\pi h_m)-\cos(21^k\pi r)}{3^k h_m}\\ &\quad+\frac{\cos(21^m\pi r+21^m\pi h_m)-\cos(21^m\pi r)}{3^m h_m} =: A+B. \end{aligned}\tag{18}$$

Exercise 6. Prove that $|A| \leqslant \frac{\pi}{6}\cdot 7^m$.

For the second term, we have by the trig identity $\cos x-\cos y = 2\sin\left(\frac{y+x}{2}\right)\sin\left(\frac{y-x}{2}\right)$ and the definition of h_m

$$\begin{aligned}|B| &= \frac{2}{3^m h_m}\left|\sin\frac{21^m\pi h_m}{2}\right|\left|\sin\left(21^m\pi r+\frac{21^m\pi h_m}{2}\right)\right|\\ &= \frac{21}{s}\cdot 7^m\left|\sin\frac{s}{21}\pi\right|\left|\sin\left(21^m\pi r+\frac{s}{21}\pi\right)\right|\\ &= 7^m\pi\frac{\left|\sin\frac{s}{21}\pi\right|}{\frac{s}{21}\pi}\left|\sin\left(21^m\pi r+\frac{s}{21}\pi\right)\right|\\ &\geqslant \frac{4}{3\sqrt{2}}\cdot 7^m\left|\sin\left(21^m\pi r+\frac{s}{21}\pi\right)\right|. \end{aligned}\tag{19}$$

Exercise 7. Prove that when $s<\frac{63}{4}$, $\frac{\left|\sin\frac{s}{21}\pi\right|}{\frac{s}{21}\pi} > \frac{4}{3\sqrt{2}\pi}$. (Hint:[2])

Now consider the expansion of r in base 21:

$$r = r_0+\frac{r_1}{21}+\frac{r_2}{21^2}+\cdots+\frac{r_m}{21^m}+\cdots. \tag{20}$$

We have

$$\sin\left(21^m\pi r+\frac{s}{21}\pi\right) = \sin\left(\frac{r_{m+1}+s}{21}\pi+\frac{r_{m+2}}{21^2}\pi+\cdots\right). \tag{21}$$

Exercise 8. Prove that there are $s_{m1}, s_{m2} < \frac{63}{4}$ such that $\sin\left(21^m\pi r+\frac{s_{m1}}{21}\pi\right) \geqslant \frac{1}{\sqrt{2}}$ and $\sin\left(21^m\pi r+\frac{s_{m2}}{21}\pi\right) \leqslant -\frac{1}{\sqrt{2}}$.

Exercise 9. Let $h_{m1} := \frac{2s_{m1}}{21^{m+1}}$, $h_{m2} := \frac{2s_{m2}}{21^{m+1}}$. Prove that

$$\lim_{m\to\infty}\left|\frac{f(r+h_{m1})-f(r)}{h_{m1}}-\frac{f(r+h_{m2})-f(r)}{h_{m2}}\right| = \infty \tag{22}$$

and conclude that f is not differentiable at r. □

Problem 2. Prove that

$$f(x) := \sum_{n=1}^{\infty} b^n\cos(a^n\pi x) \tag{23}$$

where $b\in(0,1)$ and a is an odd integer with $ab > 1+\frac{3\pi}{2}$ is continuous on $\mathbb{R}$ but nowhere differentiable.

- (Riemann's Example) Riemann proposed[3] the following function

$$g(x) := \sum_{n=1}^{\infty}\frac{\sin(n^2 x)}{n^2} \tag{24}$$

2. Monotonicity of $\frac{\sin x}{x}$.

3. There is no official record, but Weierstrass stated in a 1875 letter that he "knew" Riemann had constructed this function as early as 1861.

as a candidate for "everywhere continuous but nowhere differentiable" functions. $g(x)$ may look similar to $f(x)$ but the replacement of $\sin(n\,x)$ by $\sin(n^2\,x)$ totally changed the game. The continuity part is as trivial as that for $f(x)$, but the differentiability part is much more difficult. G. H. Hardy in 1916 prove that $g(x)$ is indeed not differentiable at x when $x/\pi\notin\mathbb{Q}$. Joseph L. Gerver[4] finally proved in 1970/1972 that $g'(x)=-\frac{1}{2}$ at all points of the form $\frac{2\,r+1}{2\,s+1}\,\pi$ where $r,\,s\in\mathbb{Z}$, and $g(x)$ is not differentiable at every other rational multiple of π. Thus the differentiability of $g(x)$ is completely understood.

4. Now at Rutgers University: http://math.camden.rutgers.edu/faculty/.