Math 118 Winter 2015 Lecture 30 (Mar. 5, 2015)

- Recall the theory of uniform convergence.

  Theorem 1. (Properties of uniformly convergent series) *Let $\sum_{n=1}^{\infty} u_n(x)$ be a infinite series of functions. Assume . Then*

  i. If $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on $[a,b]$ to $f(x)$ and each $u_n(x)$ is continuous, then $f(x)$ is continuous on $[a,b]$;

  ii. If each $u_n(x)$ is differentiable on (a,b) and

  1. $\sum_{n=1}^{\infty} u_n(x_0)$ converges for some $x_0 \in (a,b)$;

  2. $\sum_{n=1}^{\infty} u_n'(x)$ converges to $\varphi(x)$ uniformly on (a,b),

  then

  1. $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly to some $f(x)$ on (a,b),

  2. f is differentiable and $f'(x) = \varphi(x)$ on (a,b).

  iii. If $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly to $f(x)$ on $[a,b]$ and each $u_n(x)$ is integrable on $[a,b]$, then $f(x)$ is integrable on $[a,b]$ and furthermore

$$\sum_{n=1}^{\infty} \int_a^b u_n(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x. \tag{1}$$

- An example of trigonometric series.

  We discuss the continuity, integrability, and differentiability of the function

$$f(x) := \sum_{n=1}^{\infty} \frac{\sin(n\,x)}{n^2}. \tag{2}$$

1. The function is defined for all $x \in \mathbb{R}$.

   Proof. Let $x \in \mathbb{R}$ be arbitrary. Then we have

$$\left| \frac{\sin(n\,x)}{n^2} \right| \leqslant \frac{1}{n^2}. \tag{3}$$

   As $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, by Comparison Theorem we have the convergence of $\sum_{n=1}^{\infty} \frac{\sin(n\,x)}{n^2}$. □

2. The function is continuous on $\mathbb{R}$.

   Proof. We prove the convergence is uniform on $\mathbb{R}$. This follows immediately from (3), the convergence of $\sum_{n=1}^{\infty} \frac{1}{n^2}$, and Weierstrass' M-test.

Now since for each fixed n, $\frac{\sin(n\,x)}{n^2}$ is continuous on $\mathbb{R}$, $f(x)$ is also continuous on $\mathbb{R}$. □

3. The function is Riemann integrable on any compact interval $[a,b]\subset\mathbb{R}$, and furthermore

$$\int_a^b f(x)\,\mathrm{d}x = \sum_{n=1}^{\infty} \int_a^b \frac{\sin(n\,x)}{n^2}\,\mathrm{d}x. \tag{4}$$

Proof. This follows immediately from the uniform convergence we have just proved. □

4. The function is differentiable at every $x \neq 2\,k\,\pi$ $(k\in\mathbb{Z})$, and furthermore at such x,

$$f'(x) = \sum_{n=1}^{\infty} \frac{\cos(n\,x)}{n}. \tag{5}$$

Proof. Since

$$\left(\frac{\sin(n\,x)}{n^2}\right)' = \frac{\cos(n\,x)}{n}, \tag{6}$$

all we need to show is the uniform convergence of

$$\sum_{n=1}^{\infty} \frac{\cos(n\,x)}{n}. \tag{7}$$

First it is clear that the series does not converge for $x = 2\,k\,\pi$ for any $k\in\mathbb{Z}$. Thus in the following we focus on $x \neq 2\,k\,\pi$.

To prove the convergence we apply Abel's re-summation trick:

- First we obtain a good formula for

$$S_n(x) := \cos x + \cdots + \cos(n\,x). \tag{8}$$

We have

$$\begin{aligned} S_n(x) &= \frac{\sin(x/2)}{\sin(x/2)}\,[\cos x + \cdots + \cos(n\,x)] \\ &= \frac{1}{\sin(x/2)}\,[\sin(x/2)\cos x + \cdots + \sin(x/2)\cos(n\,x)] \\ &= \frac{1}{2\sin(x/2)}\left[\left(\sin\left(x+\frac{x}{2}\right) - \sin\left(x-\frac{x}{2}\right)\right) + \cdots + \left(\sin\left(n\,x+\frac{x}{2}\right) - \right.\right. \\ &\quad \left.\left.\sin\left(n\,x-\frac{x}{2}\right)\right)\right] \\ &= \frac{\sin(n\,x + x/2)}{2\sin(x/2)} - 1/2. \end{aligned} \tag{9}$$

We see that for any compact interval $[a,\,b]$ not containing $2\;k\;\pi$, there is $M = M(a,\,b)$ (that is, depending on $a,\,b$ – more precisely depending on the distance between a, b and the nearest $2\,k\,\pi$) such that

$$\forall n\in\mathbb{N},\quad \forall x\in[a,b],\qquad |S_n(x)| < M. \tag{10}$$

- Now we apply the re-summation trick. For any $m > n$, we have

$$\begin{aligned}\left|\frac{\cos((n+1)\,x)}{n+1}+\cdots+\frac{\cos(m\,x)}{m}\right| &= \left|\frac{S_{n+1}(x)-S_n(x)}{n+1}+\cdots+\frac{S_m(x)-S_{m-1}(x)}{m}\right|\\ &= \left|\frac{S_m(x)}{m}-\frac{S_n(x)}{n+1}+S_{n+1}(x)\left(\frac{1}{n+1}-\frac{1}{n+2}\right)+\cdots+S_{m-1}(x)\left(\frac{1}{m-1}-\frac{1}{m}\right)\right|\\ &\leqslant \frac{M}{m}+\frac{M}{n+1}+M\left[\left(\frac{1}{n+1}-\frac{1}{n+2}\right)+\cdots+\left(\frac{1}{m-1}-\frac{1}{m}\right)\right]\\ &= \frac{2\,M}{n+1}. \qquad (11)\end{aligned}$$

  Note that this holds for every $x \in [a, b]$.

- Finally we prove uniform convergence.

  Taking $m \to \infty$ in the above estimate, we have (denote the limit function by $\phi(x)$)

$$\forall n \in \mathbb{N}, \quad \forall x \in [a, b], \qquad |\phi(x) - S_n(x)| \leqslant \frac{2\,M}{n+1}. \qquad (12)$$

  Now let $\varepsilon > 0$ be arbitrary. Take $N > \frac{2\,M}{\varepsilon}$. Then for every $n > N$ and every $x \in [a, b]$, we have

$$|\phi(x) - S_n(x)| \leqslant \frac{2\,M}{n+1} < \frac{2\,M}{N} < \varepsilon. \qquad (13)$$

  Thus $S_n(x) \longrightarrow \phi(x)$ uniformly on $[a, b]$.

Now take any $x \neq 2\,k\,\pi$. There is $a < x < b$ such that $[a, b]$ does not contain any $2\,k\,\pi$. We see that $\sum_{n=1}^{\infty} \frac{\cos(n\,x)}{n}$ converges uniformly on $[a, b]$. Consequently $f(x)$ is differentiable on (a, b) and in particular at x. □

5. The function is not differentiable at every $x = 2\,k\,\pi$ $(k \in \mathbb{Z})$.

Proof. Again thanks to periodicity, all we need to prove is $f'(0)$ does not exist. We achieve this through proving

$$\lim_{m \to \infty} \frac{f(1/m) - f(0)}{1/m} = +\infty. \qquad (14)$$

Clearly $f(0) = 0$. We have

$$\begin{aligned}\frac{f(1/m)}{1/m} &= \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n}{m}\right)}{n^2/m}\\ &= \sum_{n=1}^{m} \frac{1}{n}\,\frac{\sin(n/m)}{n/m} + m \sum_{n=m+1}^{\infty} \frac{\sin(n/m)}{n^2}.\end{aligned}$$

We denote the two sums by A and B.

- Estimate of A.

It is easy to prove that $\frac{\sin x}{x}$ is decreasing on $(0, \pi/2)$. Thus for each term in A we have

$$\frac{n}{m} \leqslant 1 \Longrightarrow \frac{\sin(n/m)}{n/m} \geqslant \frac{\sin 1}{1}. \tag{15}$$

Therefore

$$A \geqslant c \sum_{n=1}^{m} \frac{1}{n} \tag{16}$$

where $c := (\sin 1)/1 > 0$ is a fixed constant.

- Estimate of B.

  We have

$$|B| \leqslant m \sum_{n=m+1}^{\infty} \frac{1}{n^2} < m \sum_{n=m+1}^{\infty} \frac{1}{(n-1)\,n} = m \sum_{n=m+1}^{\infty} \left[\frac{1}{n-1} - \frac{1}{n}\right] = 1. \tag{17}$$

Putting the estimates together, we have

$$\frac{f(1/m)}{1/m} > c \sum_{n=1}^{m} \frac{1}{n} - 1 \tag{18}$$

whose limit is obviously ∞ as $m \to \infty$.

Thus we have found a sequence $x_m \longrightarrow 0$ such that

$$\lim_{m \to \infty} \frac{f(x_m) - f(0)}{x_m - 0} = +\infty \tag{19}$$

and it follows that f cannot be differentiable at 0. ☐

Exercise 1. Prove that $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = +\infty$.

Exercise 2. Let $f(x) := \sum_{n=1}^{\infty} \frac{\cos(n\,x)}{n^2}$.

a) Find all $x \in \mathbb{R}$ where f is continuous, justify.

b) Find all $x \in \mathbb{R}$ where f is differentiable, justify.