Math 118 Winter 2015 Lecture 28 (Mar. 2, 2015)

- Power series.
  - Recall that a power series is a special kind of infinite series of functions where each $u_n(x) = a_n (x - x_0)^n$ with $a_n \in \mathbb{R}$ and $x_0 \in \mathbb{R}$ independent of n. Thus a power series looks like

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{1}$$

which is a shorthand for the infinite sum

$$a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \cdots \tag{2}$$

Exercise 1. Consider the power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ and $\sum_{n=0}^{\infty} a_n x^n$. Prove that

a) the former converges at c if and only if the latter converges at $c - x_0$;

b) the former converges uniformly on $[a, b]$ if and only if the latter converges uniformly on $[a - x_0, b - x_0]$.

  - Radius of convergence.

Theorem 1. *Let $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ be a power series. Define $R := \left(\limsup_{n\to\infty} |a_n|^{1/n}\right)^{-1}$. Then*

a) The power series converges for $|x - x_0| < R$ and diverges for $|x - x_0| > R$.

b) Let $0 < r < R$ be arbitrary. Then the power series converges uniformly on $[x_0 - r, x_0 + r]$.

Proof. For a) see 117 notes. We prove the new conclusion b).

Let $0 < r < R$ be arbitrary. Define $r_1 := \frac{R + r}{2}$. Then we have $0 < r < r_1 < R$. Therefore $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ converges at $x_0 + r_1$ which gives the convergence of $\sum_{n=0}^{\infty} a_n r_1^n$ and consequently $\lim_{n\to\infty} a_n r_1^n = 0$.

Exercise 2. Prove that there is $M > 0$ such that for all $n \in \mathbb{N}$, $|a_n r_1^n| \leqslant M$.

Therefore for all $n \in \mathbb{N}$, $|a_n| \leqslant \frac{M}{r_1^n}$. Now we have

$$\forall x \in [x_0 - r, x_0 + r], \qquad |a_n (x - x_0)^n| \leqslant \frac{M}{r_1^n} r^n = M \theta^n \tag{3}$$

where $\theta := \frac{r}{r_1} \in (0, 1)$.

Exercise 3. Prove that $\sum_{n=0}^{\infty} M \theta^n$ converges.

Thus by Weierstrass' theorem $\sum_{n=0}^{\infty} a_n r_1^n$ converges uniformly on $[x_0 - r, x_0 + r]$. $\square$

Exercise 4. Let $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ be a power series with radius of convergence R. Let $[a, b] \subset (x_0 - R, x_0 + R)$ be arbitrary. Then $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ converges uniformly on $[a, b]$.

- Application of uniform convergence theory.

Theorem 2. *Let $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ be a power series with radius of convergence R. Define $f(x) := \sum_{n=0}^{\infty} a_n (x - x_0)^n$. Then*

a) $f(x)$ is continuous on $(x_0 - R, x_0 + R)$.

b) *$f(x)$ is differentiable on $(x_0 - R, x_0 + R)$, and $f'(x) = \sum_{n=0}^{\infty} (n+1)\, a_{n+1}\, (x - x_0)^n$ on $(x_0 - R, x_0 + R)$.*

c) *$f(x)$ is infinitely differentiable on $(x_0 - R, x_0 + R)$.*

d) *Let $[a,\, b] \subset (x_0 - R,\, x_0 + R)$ be arbitrary. Then $f(x)$ is integrable on $[a,\, b]$ and furthermore*

$$\int_a^b f(x)\, \mathrm{d}x = \sum_{n=0}^{\infty} a_n \int_a^b (x - x_0)^n\, \mathrm{d}x. \tag{4}$$

Proof. We prove a), b) and leave c), d) as exercises.

a) Let $x \in (x_0 - R, x_0 + R)$ be arbitrary. All we need to prove is that $f(x)$ is continuous at x. As $\delta := \min\{|(x_0 + R) - x|, |(x_0 - R) - x|\} > 0$, we have

$$x \in (x_0 - r, x_0 + r) \subset [x_0 - r, x_0 + r] \subset (x_0 - R, x_0 + R) \tag{5}$$

for $r := R - \frac{\delta}{2}$. By Theorem 1 $\sum_{n=0}^{\infty} a_n\, (x - x_0)^n$ converges uniformly on $[x_0 - r, x_0 + r]$. As for every n, $a_n\, (x - x_0)^n$ is contiuous on $[x_0 - r, x_0 + r]$, we have $f(x)$ continuous on $[x_0 - r, x_0 + r]$ and thus in particular continuous at x.

b) Let $x \in (x_0 - R, x_0 + R)$ be arbitrary. Let r be defined as in a). Then $\sum_{n=0}^{\infty} a_n\, (x - x_0)^n$ converges uniformly on $[x_0 - r, x_0 + r]$.

Now consider

$$\sum_{n=0}^{\infty} (a_n\, (x - x_0)^n)' = \sum_{n=0}^{\infty} n\, a_n\, (x - x_0)^{n-1} = \sum_{n=0}^{\infty} (n+1)\, a_{n+1}\, (x - x_0)^n. \tag{6}$$

As this is again a power series, we calculate its radius of convergence R_1.

Exercise 5. Prove that $R_1 = R$.

As $R_1 = R$, $\sum_{n=0}^{\infty} (a_n\, (x - x_0)^n)'$ also converges uniformly on $[x_0 - r, x_0 + r]$. Thus $f(x)$ is differentiable on $(x_0 - r, x_0 + r)$ and in particular differentiable at x. □

Exercise 6. Prove c) and d) of Theorem 2.