MATH 118 WINTER 2015 LECTURE 27 (FEB. 27, 2015)

- Continuity, integrability, and differentiability under uniform convergence.
  - Continuity.

    THEOREM 1. *Let $f_n(x)$ be continuous on $[a, b]$ for every n. Assume $f_n(x) \to f(x)$ uniformly. Then $f(x)$ is continuous on $[a, b]$.*

    Proof. Let $\varepsilon > 0$ and $x_0 \in [a, b]$ be arbitrary. Since $f_n(x) \longrightarrow f(x)$ uniformly, there is $n_0 \in \mathbb{N}$ such that $\forall x \in [a, b]$, $|f_{n_0}(x) - f(x)| < \varepsilon/3$.

    Now since $f_{n_0}(x)$ is continuous, there is $\delta > 0$ such that

    $$\forall |x - x_0| < \delta \Longrightarrow |f_{n_0}(x) - f_{n_0}(x_0)| < \varepsilon/3. \tag{1}$$

    Thus for the same δ, we have for every $|x - x_0| < \delta$,

    $$\begin{aligned} |f(x) - f(x_0)| &\leqslant |f(x) - f_{n_0}(x)| \\ &\quad + |f_{n_0}(x) - f_{n_0}(x_0)| \\ &\quad + |f_{n_0}(x_0) - f(x_0)| \\ &< \varepsilon. \end{aligned} \tag{2}$$

    Thus ends the proof. □

  - Integrability.

    THEOREM 2. *Let $f_n(x)$ be Riemann integrable on $[a, b]$ for every n. Assume $f_n(x) \to f(x)$ uniformly. Then $f(x)$ is Riemann integrable on $[a, b]$ and*

    $$\lim_{n \to \infty} \int_a^b f_n(x)\, \mathrm{d}x = \int_a^b f(x)\, \mathrm{d}x. \tag{3}$$

    Proof. Let $\varepsilon > 0$ be arbitrary. To show the integrability of f all we need is to find a partiaion P of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$.

    Since $f_n(x) \longrightarrow f(x)$ uniformly on $[a, b]$, there is $n_0 \in \mathbb{N}$ such that

    $$\forall x \in [a, b], \qquad |f_{n_0}(x) - f(x)| < \frac{\varepsilon}{3\,(b - a)}. \tag{4}$$

    Since $f_{n_0}(x)$ is Riemann integrable on $[a, b]$, there is a partition $P = \{a = x_0 < x_1 < \cdots < x_m = b\}$ such that

    $$U(f_{n_0}, P) - L(f_{n_0}, P) < \frac{\varepsilon}{3}. \tag{5}$$

    Now we have

    $$\begin{aligned} U(f, P) &= \sum_{k=1}^m \left(\sup_{x \in [x_{k-1}, x_k]} f(x) \right) (x_k - x_{k-1}) \\ &= \sum_{k=1}^m \left(\sup_{x \in [x_{k-1}, x_k]} [f_{n_0}(x) + f(x) - f_{n_0}(x)] \right) (x_k - x_{k-1}) \\ &\leqslant \sum_{k=1}^m \left(\sup_{x \in [x_{k-1}, x_k]} f_{n_0}(x) + \sup_{x \in [x_{k-1}, x_k]} |f(x) - f_{n_0}(x)| \right) (x_k - x_{k-1}) \\ &< \sum_{k=1}^m \left(\sup_{x \in [x_{k-1}, x_k]} f_{n_0}(x) + \frac{\varepsilon}{3\,(b - a)} \right) (x_k - x_{k-1}) \\ &= U(f_{n_0}, P) + \frac{\varepsilon}{3}. \end{aligned} \tag{6}$$

Similarly we have

$$L(f,P) > L(f_{n_0},P) - \frac{\varepsilon}{3}. \tag{7}$$

Therefore $U(f,P) - L(f,P) < \left[U(f_{n_0},P) + \frac{\varepsilon}{3}\right] - \left[L(f_{n_0},P) - \frac{\varepsilon}{3}\right] < \varepsilon$. Thus ends the proof. □

- Differentiability.

THEOREM 3. *Let $f_n(x)$ be differentiable on $[a,b]$ and satisfies:*

i. There is $x_0 \in E$ such that $f_n(x_0)$ convergens;

ii. $f_n'(x)$ converges uniformly to some function $\varphi(x)$ on $[a,b]$;

Then

a) $f_n(x)$ converges uniformly to some function $f(x)$ on $[a,b]$;

b) $f'(x) = \varphi(x)$ on $[a,b]$.

Proof.

a) First we show that $\forall x \in [a,\, b]$, $f_n(x)$ converges. It suffices to show that the sequence $f_n(x)$ is Cauchy.

Let $\varepsilon > 0$ be arbitrary. Take $N_0 \in \mathbb{N}$ such that for all $n > N_0$,

$$\forall x \in [a,b], \qquad |f_n'(x) - \varphi(x)| < \frac{\varepsilon}{4\,(b-a)}. \tag{8}$$

On the other hand, since $f_n(x_0) \longrightarrow f(x_0)$, there is $N_1 \in \mathbb{N}$ such that for all m, $n > N_1$,

$$|f_n(x_0) - f_m(x_0)| < \frac{\varepsilon}{2}. \tag{9}$$

Now take $N = \max\{N_0, N_1\}$. We have, for any $m, n > N$,

$$\begin{aligned} |f_n(x) - f_m(x)| &\leqslant |f_n(x_0) - f_m(x_0)| \\ &\quad + |(f_n(x) - f_n(x_0)) - (f_m(x) - f_m(x_0))| \\ &< \frac{\varepsilon}{2} + |f_n'(\xi) - f_m'(\xi)|\,|b-a| < \varepsilon. \end{aligned} \tag{10}$$

Thus there is $f(x)$ defined on $[a,\, b]$ such that $f_n(x) \longrightarrow f(x)$. The proof of uniformity is left as exercise.

b) We consider

$$\frac{f_m(x) - f_m(x_0)}{x - x_0} - \frac{f_n(x) - f_n(x_0)}{x - x_0} = f_m'(\xi) - f_n'(\xi). \tag{11}$$

Thus we have

$$\frac{f_n(x) - f_n(x_0)}{x - x_0} \longrightarrow \frac{f(x) - f(x_0)}{x - x_0} \tag{12}$$

uniformly in x. Now define

$$F_n(x) := \begin{cases} \dfrac{f_n(x) - f_n(x_0)}{x - x_0} & x \neq x_0 \\ f_n'(x_0) & x = x_0 \end{cases}; \; F(x) := \begin{cases} \dfrac{f(x) - f(x_0)}{x - x_0} & x \neq x_0 \\ \varphi(x_0) & x = x_0 \end{cases} \tag{13}$$

We see that $\lim_{n\to\infty} F_n(x) = F(x)$ for all $x \in [a, b]$. As each $F_n(x)$ is continuous,

Exercise 1. Prove that each $F_n(x)$ is continuous.

if we can prove $F_n(x) \longrightarrow F(x)$ uniformly on $[a, b]$, it would follow that $F(x)$ is continuous and consequently $f'(x_0) = \varphi(x_0)$.

Exercise 2. Prove that if $F(x)$ is continuous then $f(x)$ is differentiable at x_0 and $f'(x_0) = \varphi(x_0)$.

To prove uniform convergence of $F_n(x)$, we prove that it is "uniformly Cauchy". Let $\varepsilon > 0$ be arbitrary. Since $f_n'(x)$ converges uniformly on $[a, b]$, there is $N \in \mathbb{N}$ such that for all $m > n > N$,

$$\sup_{x\in[a,b]} |f_m'(x) - f_n'(x)| < \varepsilon. \tag{14}$$

Now consider $F_m(x) - F_n(x)$.

- Case 1. $x = x_0$. We have

$$|F_m(x_0) - F_n(x_0)| = |f_m'(x_0) - f_n'(x_0)| < \varepsilon. \tag{15}$$

- Case 2. $x \neq x_0$. We have

$$\begin{aligned} |F_m(x) - F_n(x)| &= \left| \frac{f_m(x) - f_m(x_0)}{x - x_0} - \frac{f_n(x) - f_n(x_0)}{x - x_0} \right| \\ &= \left| \frac{h(x) - h(x_0)}{x - x_0} \right| \\ &\quad (h(x) := f_m(x) - f_n(x)) \\ &= |h'(c)| \text{ for some } c \in (x_0, x) \subseteq (a, b) \\ &= |f_m'(c) - f_n'(c)| < \varepsilon. \end{aligned} \tag{16}$$

Therefore for all $m > n > N$,

$$\forall x \in [a, b], \qquad |F_m(x) - F_n(x)| < \varepsilon \tag{17}$$

and uniform convergence follows. □

- Properties of Uniformly Convergent Infinite Series of Functions.

Theorem 4. (Properties of uniformly convergent series) *Let $\sum_{n=1}^{\infty} u_n(x)$ be a infinite series of functions. Assume . Then*

i. If $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly to $f(x)$ and each $u_n(x)$ is continuous, then $f(x)$ is continuous;

ii. If each $u_n(x)$ is differentiable and

   1. $\sum_{n=1}^{\infty} u_n(x_0)$ converges for some x_0;

   2. $\sum_{n=1}^{\infty} u_n'(x)$ converges to $\varphi(x)$ uniformly,

   then

   1. $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly to some $f(x)$,

   2. f is differentiable and $f'(x) = \varphi(x)$.

iii. *If* $\sum_{n=1}^{\infty} u_n(x)$ *converges uniformly to* $f(x)$ *and each* $u_n(x)$ *is integrable on* $[a, b]$*, then* $f(x)$ *is integrable on* $[a, b]$ *and furthermore*

$$\sum_{n=1}^{\infty} \int_a^b u_n(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x. \tag{18}$$

Exercise 3. Prove the above theorem.

Example 5. Let $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{3^n} \cos(n\,\pi\,x^2)$. Calculate $\lim_{x\to 1} f(x)$.

By the above theorem $f(x)$ is continuous. Thus

$$\lim_{x\to 1} f(x) = f(1) = \frac{3}{4}. \tag{19}$$