Math 118 Winter 2015 Lecture 26 (Feb. 26, 2015)

- An example.

Example 1. Consider $\sum_{n=0}^{\infty} x^n$ on $(0,1)$.

a) Does it converge?

b) If it does, is the convergence uniform?

Solution.

a) Yes. Let $x \in (0,1)$ be arbitrary.

Exercise 1. Prove that $1 + x + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}$.

We have $\lim_{n \to \infty} (1 + x + \cdots + x^n) = \lim_{n \to \infty} \frac{1 - x^{n+1}}{1 - x} = \frac{1}{1 - x}$ which is finite when $x \in (0,1)$. Therefore $\sum_{n=0}^{\infty} x^n$ converges on $(0,1)$ and the sum is $\frac{1}{1 - x}$.

b) No. Let $N \in \mathbb{N}$ be arbitrary. Set $n = 2\,N > N$ and $x_n := 1 - \frac{1}{n+2}$. Then we have $0 < 1 + x_n + \cdots + x_n^n < n + 1$ and $\frac{1}{1 - x_n} = n + 2$.

$$\left| (1 + x + \cdots + x^n) - \frac{1}{1 - x_n} \right| \geqslant 1. \tag{1}$$

- Checking uniform convergence.

  - For $f_n(x) \longrightarrow f(x)$.

Theorem 2. (Cauchy criterion) *$f_n(x)$ converges uniformly to $f(x)$ on $[a,b]$ if and only if*

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ \forall m > n > N, \forall x \in [a,b], \qquad |f_m(x) - f_n(x)| < \varepsilon. \tag{2}$$

Exercise 2. Prove Theorem 2.

Exercise 3. Prove that $f_n(x)$ converges uniformly to $f(x)$ on $[a,b]$ if and only if

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ \forall m > n > N, \qquad \sup_{x \in [a,b]} |f_m(x) - f_n(x)| < \varepsilon. \tag{3}$$

Theorem 3. *$f_n(x)$ converges uniformly to $f(x)$ on $[a,b]$ if and only if $\lim_{n \to \infty} M_n = 0$ where $M_n := \sup_{x \in [a,b]} |f_n(x) - f(x)|$.*

Proof. We prove "if" and then "only if".

– If. Assume $\lim_{n \to \infty} M_n = 0$ where $M_n := \sup_{x \in [a,b]} |f_n(x) - f(x)|$.
Let $\varepsilon > 0$ be arbitrary. As $\lim_{n \to \infty} M_n = 0$ there is $N_1 \in \mathbb{N}$ such that $|M_n| < \varepsilon$ for all $n > N_1$.
Take $N = N_1$. Then for every $n > N$, we have

$$\forall x \in [a,b], \qquad |f_n(x) - f(x)| \leqslant M_n < \varepsilon. \tag{4}$$

– Only if. Assume $f_n(x)$ converges uniformly to $f(x)$ on $[a,b]$.
Let $\varepsilon > 0$ be arbitrary. As $f_n(x)$ converges uniformly to $f(x)$ on $[a,b]$, there is $N_1 \in \mathbb{N}$ such that for all $n > N_1$,

$$\forall x \in [a,b], \qquad |f_n(x) - f(x)| < \frac{\varepsilon}{2}. \tag{5}$$

Now set $N = N_1$. For every $n > N$, we have $0 \leqslant M_n = \sup_{x \in [a,b]} |f_n(x) - f(x)| \leqslant \frac{\varepsilon}{2} < \varepsilon$. Thus $\lim_{n \to \infty} M_n = 0$. □

Example 4. Prove that the convergence is uniform on $(0, \infty)$ for $\lim_{n \to \infty} x\, e^{-nx} = 0$.

Proof. We calculate

$$(x\, e^{-nx})' = (1 - n\, x)\, e^{-nx} \tag{6}$$

which is positive for $x < 1/n$ and negative for $x > 1/n$. Therefore

$$\sup_{x \in (0,\infty)} x\, e^{-nx} = \frac{1}{n}\, e^{-n\left(\frac{1}{n}\right)} = \frac{1}{n\, e}. \tag{7}$$

As $\lim_{n \to \infty} \frac{1}{n\, e} = 0$ the convergence of $x\, e^{-nx}$ to 0 is uniform. □

- For $\sum_{n=1}^{\infty} u_n(x)$.

Exercise 4. Prove that $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on $[a, b]$ if and only if

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall m > n > N, \forall x \in [a, b], \qquad \left| \sum_{k=n+1}^{m} u_k(x) \right| < \varepsilon. \tag{8}$$

PROPOSITION 5. *If $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on $[a,\ b]$, then $u_n(x) \longrightarrow 0$ uniformly on $[a, b]$.*

Proof. Let $\varepsilon > 0$ be arbitrary. As $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly there is $N_1 \in \mathbb{N}$ such that for all $m > n > N_1$,

$$\forall x \in [a, b], \qquad \left| \sum_{k=n+1}^{m} u_k(x) \right| < \varepsilon. \tag{9}$$

In particular, setting $m = n + 1$ we have

$$\forall x \in [a, b], \qquad |u_{n+1}(x)| < \varepsilon. \tag{10}$$

Now set $N = N_1 + 1$. For every $n > N$, we have $n - 1 > N_1$ and therefore

$$\forall x \in [a, b], \qquad |u_n(x)| = |u_{n-1+1}(x)| < \varepsilon. \tag{11}$$

Thus ends the proof. □

Remark 6. In practice we often apply the contrapositive: If $u_n(x)$ does not converge to 0 uniformly on $[a, b]$, then $\sum_{n=1}^{\infty} u_n(x)$ does not converge uniformly.

Example 7. $\sum_{n=1}^{\infty} n\, x\, e^{-nx}$ does not converge uniformly on $(0, \infty)$.

Proof. Let $u_n(x) := n\, x\, e^{-nx}$. Then we have

$$M_n := \sup_{x \in (0,\infty)} |u_n(x) - 0| \geqslant \left| u_n\!\left(\frac{1}{n}\right) - 0 \right| = e^{-1}. \tag{12}$$

Therefore $\lim_{n \to \infty} M_n = 0$ does not hold, which means $u_n(x)$ does not uniformly converge to 0 on $(0, \infty)$, consequently $\sum_{n=1}^{\infty} n\, x\, e^{-nx}$ does not converge uniformly on $(0, \infty)$. □

Exercise 5. Does $\sum_{n=1}^{\infty} n\, x\, e^{-n x}$ converge on $(0, \infty)$?

THEOREM 8. (WEIERSTRASS) *If there is* $\{a_n\}$ *such that*

i. $\forall x \in [a, b], \forall n \in \mathbb{N}, |u_n(x)| \leqslant a_n$;

ii. $\sum_{n=1}^{\infty} a_n$ *converges,*

then $\sum_{n=1}^{\infty} u_n(x)$ *converges uniformly on* $[a, b]$.

Proof. Let $\varepsilon > 0$ be arbitrary. As $\sum_{n=1}^{\infty} a_n$ converges, there is $N_1 \in \mathbb{N}$ such that for all $m > n > N_1$, $\left|\sum_{k=n+1}^{m} a_n\right| < \varepsilon$.

Now take $N = N_1$. By assumption

$$\forall x \in [a, b], \qquad \left|\sum_{k=n+1}^{m} u_k(x)\right| \leqslant \sum_{k=n+1}^{m} |u_k(x)| \leqslant \sum_{k=n+1}^{m} a_k < \varepsilon. \tag{13}$$

Thus ends the proof. □

Exercise 6. Find $u_n(x)$ such that $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on $[a, b]$ but there is no $\{a_n\}$ satisfying $\forall x \in [a, b]$, $\forall n \in \mathbb{N}$, $|u_n(x)| \leqslant a_n$ and $\sum_{n=1}^{\infty} a_n$ converges.

Example 9. Consider $\sum_{n=1}^{\infty} \frac{\sin n\, x}{n^2}$. We have

$$\forall x \in \mathbb{R}, \qquad \left|\frac{\sin n\, x}{n^2}\right| \leqslant \frac{1}{n^2}. \tag{14}$$

As $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, $\sum_{n=1}^{\infty} \frac{\sin n\, x}{n^2}$ converges uniformly on $\mathbb{R}$.