Math 118 Winter 2015 Lecture 25 (Feb. 25, 2015)

- One more example.

Example 1. Study $\sum_{n=1}^{\infty} \frac{\sin n x}{n}$.

- First notice that $\left|\frac{\sin n x}{n}\right| \leqslant \frac{1}{n}$ does not lead to any conclusion as $\sum_{n=1}^{\infty} \frac{1}{n} = +\infty$.

  Remark. Note that for most x, there is no $p > 1$ such that $\left|\frac{\sin n x}{n}\right| \leqslant \frac{1}{n^p}$ holds for all $n \in \mathbb{N}$ as $\limsup_{n \to \infty} |\sin(n x)| = 1$.

- To be able to deal with $\sum_{n=1}^{\infty} \frac{\sin n x}{n}$, we need the technique of "Abel's resummation":

$$\begin{aligned}\sum_{k=1}^{n} a_k b_k &= A_1 b_1 + (A_2 - A_1) b_2 + \cdots + (A_n - A_{n-1}) b_n \\ &= A_1 (b_1 - b_2) + A_2 (b_2 - b_3) + \cdots + A_{n-1} (b_{n-1} - b_n) + A_n b_n \\ &= \sum_{k=1}^{n-1} A_k (b_k - b_{k+1}) + A_n b_n \end{aligned} \tag{1}$$

  where

$$A_k := a_1 + a_2 + \cdots + a_k. \tag{2}$$

- Now let $x \in \mathbb{R}$ be arbitrary. Set $a_n := \sin n x$ and $b_n = \frac{1}{n}$. We have following (1)

$$S_n := \sum_{k=1}^{n} \frac{\sin k x}{k} = \sum_{k=1}^{n-1} \frac{A_k}{k (k+1)} + \frac{A_n}{n}. \tag{3}$$

  We will try to prove using (3) that $\{S_n\}$ is Cauchy.

- First notice that when $x = 2 m \pi$ for some $m \in \mathbb{Z}$, $S_n = 0$ for all n and is therefore Cauchy.

- Next we claim that, if there is $M > 0$ such that $|A_n| \leqslant M$ for all $n \in \mathbb{N}$, then $S_n = \sum_{k=1}^{n-1} \frac{A_k}{k (k+1)} + \frac{A_n}{n}$ is Cauchy.

  Proof. Let $\varepsilon > 0$ be arbitrary. Set $N > \frac{2 M}{\varepsilon}$. Then for every $m > n > N$, we have

$$\begin{aligned}|S_m - S_n| &= \left|\sum_{k=n}^{m-1} \frac{A_k}{k (k+1)} + \frac{A_m}{m} - \frac{A_n}{n}\right| \\ &\leqslant \sum_{k=n}^{m-1} \frac{|A_k|}{k (k+1)} + \frac{|A_m|}{m} + \frac{|A_n|}{n} \\ &\leqslant M \left[\sum_{k=n}^{m-1} \left(\frac{1}{k} - \frac{1}{k+1}\right) + \frac{1}{m} + \frac{1}{n}\right] \\ &= \frac{2 M}{n} < \frac{2 M}{N} < \varepsilon. \end{aligned} \tag{4}$$

  Thus ends the proof. □

- Finally we prove that, for any $x \neq 2 m \pi$, there is $M > 0$ such that for all $n \in \mathbb{N}$, $|A_n| = |\sin x + \cdots + \sin n x| \leqslant M$.

We have

$$\left(\sin\frac{x}{2}\right) A_n = \sin x \sin\frac{x}{2} + \sin 2x \sin\frac{x}{2} + \cdots + \sin n x \sin\frac{x}{2}$$
$$= \frac{1}{2}\left[\cos\frac{x}{2} - \cos\frac{3x}{2} + \cos\frac{3x}{2} - \cos\frac{5x}{2} + \cdots + \cos\left(n - \frac{1}{2}\right) x - \cos\left(n + \frac{1}{2}\right) x\right]$$
$$= \frac{1}{2}\left[\cos\frac{x}{2} - \cos\left(n + \frac{1}{2}\right) x\right]. \qquad (5)$$

This gives

$$\forall n \in \mathbb{N}, \qquad |A_n| \leqslant \frac{1}{\left|\sin\frac{x}{2}\right|} =: M. \qquad (6)$$

Note that when $x \neq 2 m \pi$, $\sin\frac{x}{2} \neq 0$ so M is indeed a finite number.

- Uniform convergence.
  - Motivation.

    Let $f_n(x)$ converge to $f(x)$ on $[a, b]$. Is it possible to draw conclusion about continuity, differentiability, integrability of f from those of f_n?

Example 2. Consider the following.

1. For every $n \in \mathbb{N}$, $f_n(x) = x^n$ is continuous on $[0,1]$. But $f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 1 & x = 1 \\ 0 & x \in [0,1) \end{cases}$ is not continuous on $[0,1]$.
2. For every $n \in \mathbb{N}$, $f_n(x) = n\, x\, (1 - x^2)^n$ is integrable on $[0, 1]$. $f(x) = \lim_{n\to\infty} f_n(x) = 0$ is also integrable on $[0,1]$. But

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, \mathrm{d}x = \frac{1}{2} \neq 0 = \int_0^1 f(x)\, \mathrm{d}x. \qquad (7)$$

   Exercise 1. Prove $\lim_{n\to\infty} \int_0^1 f_n(x)\, \mathrm{d}x = \frac{1}{2}$.

3. For every $n \in \mathbb{N}$, $f_n(x) = \lim_{m\to\infty} (\cos(n!\, \pi\, x))^{2m} = \begin{cases} 1 & n!\, \pi\, x \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$ is integrable on $[0,1]$. But $\lim_{n\to\infty} f_n(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ is not Riemann integrable on $[0,1]$.

   Exercise 2. Prove $\lim_{m\to\infty} (\cos(n!\, \pi\, x))^{2m} = \begin{cases} 1 & n!\, \pi\, x \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$.

  - To be able to draw conclusion about continuity, differentiability, integrability of f from those of f_n, we need a stronger kind of convergence.
  - Uniform convergence of function sequences.

DEFINITION 3. (UNIFORM CONVERGENCE) *We say $f_n(x)$ converge to $f(x)$ uniformly on $[a, b]$ if and only if*

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ \forall x \in [a, b],\ \forall n > N, \qquad |f_n(x) - f(x)| < \varepsilon. \qquad (8)$$

Example 4. Prove that $f_n(x) = \frac{x}{1 + n^2 x}$ converge to 0 uniformly on $(0, \infty)$.

Proof. Let $\varepsilon > 0$ be arbitrary. Set $N > \varepsilon^{-1/2}$. Then for every $x \in (0, \infty)$ and every $n > N$ we have

$$|f_n(x) - 0| = \frac{x}{1 + n^2 x} < \frac{1}{n^2} < \frac{1}{N^2} < \varepsilon. \tag{9}$$

Thus ends the proof. □

Example 5. Prove that $f_n(x) = x^n$ does not uniformly converge to 0 on $(0, 1)$.

Proof. First we write down the working negation: f_n does not uniformly converge to f on $[a, b]$ if and only if

$$\exists \varepsilon_0 > 0,\ \forall N \in \mathbb{N},\ \exists x \in [a, b],\ \exists n > N, \qquad |f_n(x) - f(x)| \geqslant \varepsilon_0. \tag{10}$$

Now let $\varepsilon_0 = \frac{1}{2}$. Let $N \in \mathbb{N}$ be arbitrary. Now set $n = 2N > N$, $x = \left(\frac{1}{2}\right)^{1/2N} \in (0, 1)$, we have

$$|x^n - 0| = \frac{1}{2} \geqslant \varepsilon_0. \tag{11}$$

Thus ends the proof. □