Math 118 Winter 2015 Lecture 24 (Feb. 23, 2015)

- Please review the concepts of limit of sequences and convergence of infinite series.
- Motivation
  - Solving algebraic equations (Newton's method)

Example 1. Solve $y^3 - 2\,y - 5 = 0$.

Solution. First we try integers. The best is $y = 2$.

Exercise 1. Prove that $y = 2$ minimizes $|y^3 - 2\,y - 5|$ among all integers.

So write $y = 2 + p$. We obtain the equation for p:

$$p^3 + 6\,p^2 + 10\,p - 1 = 0. \tag{1}$$

Assuming that p is small, we conclude $10\,p - 1 \approx 0$. So write $p = \frac{1}{10} + q$. Substituting into the equation for p, we obtain the equation for q:

$$q^3 + 6.3\,q^2 + 11.23\,q + 0.061 = 0. \tag{2}$$

From $11.23\ q\ +\ 0.061\ \approx\ 0$ we conclude $q = -0.0054 + r$, which leads to $r = -0.00004852 + s$, ...

Thus we obtained an infinite series representation of the solution:

$$y = 2 + 0.1 - 0.0054 - 0.00004852 + \cdots \tag{3}$$

Remark. Note that, also there are formulas for roots of cubic polynomials, the formula reads

$$y = \sqrt[3]{\cdots\sqrt{\cdots}} - \sqrt[3]{\cdots\sqrt{\cdots}} \tag{4}$$

and is much inferior to (3) if we would like to have a reasonably accurate numerical value of the solution.

Example 2. Solve $y = y(x)$ for $y^3 + x\,y + y - x^3 - 2 = 0$.

Solution. Newton first assumed x is small. This gives $y^3 + y - 2 \approx 0$ and gives $y(x) = 1 + p(x)$. The equation for p reads

$$p^3 + 3\,p^2 + 4\,p + x\,p + x - x^3 = 0. \tag{5}$$

As obviously $p(0) = 0$ we see that for x small, $p \approx t\,x$ for some number t and is thus as small as x.

Exercise 2. Why is this reasonable?

This means

$$4\,p + x \approx 0 \Longrightarrow p(x) = -\frac{x}{4} + q(x). \tag{6}$$

Substituting this into the p-equation and picking the lowest order terms, we have $q(x) \approx \frac{1}{64}\,x^2 + r(x)$, ...

Finally we represent the solution as an infinite series of functions

$$y(x) = 1 - \frac{1}{4}\,x + \frac{1}{64}\,x^2 + \frac{131}{512}\,x^3 + \cdots \tag{7}$$

- Solving differential equations (Power series method)

  Example 3. Solve $y' = y,\ y(0) = 1$.

  Solution. We write $y(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ and substitute into the equation to obtain

  $$a_1 + 2 a_2 x + 3 a_3 x^2 + \cdots = a_0 + a_1 x + a_2 x^2 + \cdots, \qquad a_0 = 1. \tag{8}$$

  From this we conclude

  $$a_0 = 1, a_1 = 1, a_2 = \frac{1}{2}, a_3 = \frac{1}{6}, \ldots, a_n = \frac{1}{n!}, \ldots \tag{9}$$

  Thus the solution is

  $$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{10}$$

  Exercise 3. Solve $y' = x\, y,\ y(0) = 2$.

- Limit of functions.
  - Definition.

    Let $f_n(x): [a, b] \mapsto \mathbb{R}$ be a sequence of functions. We say $f(x) = \lim_{n\to\infty} f_n(x)$ if and only if

    $$\forall x \in [a, b], \qquad \lim_{n\to\infty} f_n(x) = f(x). \tag{11}$$

    Note that once the arbitrary $x \in [a, b]$ is chosen, $f_n(x)$ is just a number for every n.
  - Examples.

    Example 4. Let $f_n(x) = x^n$, then on $[0, 1]$, $\lim_{n\to\infty} f_n(x) = \begin{cases} 1 & x = 1 \\ 0 & x \in [0, 1) \end{cases}$.

    Example 5. Let $f_n(x) = \frac{\sin n x}{\sqrt{n}}$. For every $x \in \mathbb{R}$, we have $-\frac{1}{\sqrt{n}} \leqslant \frac{\sin n x}{\sqrt{n}} \leqslant \frac{1}{\sqrt{n}}$ and it follows from Squeeze that $\lim_{n\to\infty} \frac{\sin n x}{\sqrt{n}} = 0$. Thus

    $$\lim_{n\to\infty} \frac{\sin n x}{\sqrt{n}} = 0 \tag{12}$$

    over $\mathbb{R}$.

    Example 6. Let $f_n(x) = n\, x\, (1 - x^2)^n$. We have $\lim_{n\to\infty} f_n(x) = 0$ on $[0, 1]$.

- Infinite series of functions.
  - Definition.

    Let $u_n(x): [a, b] \mapsto \mathbb{R}$. Then $\sum_{n=1}^{\infty} u_n(x)$ is defined as $\lim_{n\to\infty} S_n(x)$ where $S_n(x) := u_1(x) + \cdots + u_n(x)$.
  - Examples.

    Example 7. $\sum_{n=1}^{\infty} \frac{x^n}{n!} = e^x$.

    Example 8. Consider $\sum_{n=1}^{\infty} \frac{\sin n x}{n^2}$.

    In contrast to Example 7, we do not have an elementary function representation of this infinite sum. However we still can prove that $f(x) := \sum_{n=1}^{\infty} \frac{\sin n x}{n^2}$ is defined at every $x \in \mathbb{R}$.

    Let $x \in \mathbb{R}$ be arbitrary. Then $\left| \frac{\sin n x}{n^2} \right| \leqslant \frac{1}{n^2}$ for all $n \in \mathbb{N}$. As $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so does $\sum_{n=1}^{\infty} \frac{\sin n x}{n^2}$.