MATH 118 WINTER 2015 LECTURE 23 (FEB. 13, 2015)

- Tests.

THEOREM 1. (DIRICHLET) *Let* $f, g: [0, \infty) \mapsto \mathbb{R}$ *be integrable on* $[0, d]$ *for every* $d > 0$*. Assume*

i. $\exists M > 0, \forall d \in \mathbb{R}, \left| \int_0^d f(t)\,\mathrm{d}t \right| \leqslant M$;

ii. g *is monotone with* $\lim_{x \to \infty} g(x) = 0$.

Then $f g$ *is improperly integrable on* $(0, \infty)$.

The proof of Theorem 1 is technical and we do not present it here. Instead we prove the following weaker version.

THEOREM 2. *Let* $f, g: [0, \infty) \mapsto \mathbb{R}$ *be continuous on* $[0, d]$ *for every* $d > 0$*. Assume*

i. $\exists M > 0, \forall d \in \mathbb{R}, \left| \int_0^d f(t)\,\mathrm{d}t \right| \leqslant M$;

ii. g *is monotone with* $\lim_{x \to \infty} g(x) = 0$;

iii. g *is differentiable with* g' *integrable on* $[0, d]$ *for every* $d > 0$.

Then $f g$ *is improperly integrable on* $(0, \infty)$.

Proof. By assumption $f g$ is integrable on $[0, d]$ for every $d > 0$. Denote $F(x) := \int_0^x f(t)\,\mathrm{d}t$. Then by FTC2 we have $F'(x) = f(x)$ and by assumption $|F(x)| \leqslant M$ for all $x > 0$.

Now we calculate

$$\begin{aligned} \int_0^d f(x)\, g(x)\,\mathrm{d}x &= \int_0^d g(x)\,\mathrm{d}F(x) \\ &= g(d)\, F(d) - g(0)\, F(0) - \int_0^d F(x)\, g'(x)\,\mathrm{d}x. \end{aligned} \tag{1}$$

Exercise 1. Prove that $\lim_{d \to \infty} g(d)\, F(d) = 0$. (Hint:[1])

Thus all we need to show is that $A(d) := \int_0^d F(x)\, g'(x)\,\mathrm{d}x$ is Cauchy. This follows from the calculation

$$\begin{aligned} |A(d_2) - A(d_1)| &= \left| \int_{d_1}^{d_2} F(x)\, g'(x)\,\mathrm{d}x \right| \\ &\leqslant M \int_{d_1}^{d_2} |g'(x)|\,\mathrm{d}x \\ &= M \left| \int_{d_1}^{d_2} g'(x)\,\mathrm{d}x \right| \qquad (2) \\ &= M\, |g(d_2) - g(d_1)|. \qquad (3) \end{aligned}$$

Exercise 2. Explain why (2) holds and write down the detailed proof of the claim: $A(d)$ is Cauchy.

1. Squeeze.

Therefore $\lim_{d\to\infty}\int_0^d f(x)\,g(x)\,\mathrm{d}x$ exists and is finite and the conclusion follows. $\square$

THEOREM 3. (ABEL) *Let $f, g\colon [0,\infty)\mapsto\mathbb{R}$ be integrable on $[0,d]$ for every $d>0$. Assume*

i. $f(x)$ is improperly integrable on $(0,\infty)$;

ii. g is monotone and bounded.

Then $f g$ is improperly integrable on $(0,\infty)$.

Proof. As $f(x)$ is improperly integrable on $(0,\infty)$, there is $A\in\mathbb{R}$ such that

$$\lim_{d\to\infty}\int_0^d f(x)\,\mathrm{d}x = A. \tag{4}$$

Thus there is $d_0>0$ such that for all $d>d_0$, $\left|\int_0^d f(x)\,\mathrm{d}x - A\right| < 1$. Now since $f(x)$ is integrable on $[0,\,d_0]$, there is $K>0$ such that $|f(x)|\leqslant K$ for all $x\in[0,\,d_0]$. Now taking $M := \max\{K\,d_0, |A|+1\}$ we easily see that $\left|\int_0^d f(x)\,\mathrm{d}x\right| \leqslant M$ for all $d\in(0,\infty)$.

Since g is monotone and bounded, there is $p\in\mathbb{R}$ such that $\lim_{x\to\infty} g(x)=p$.

Now apply Theorem 1 to $f(x)$ and $g(x)-p$, we see that $f(x)\,[g(x)-p]$ is improperly integrable on $(0,\infty)$.

Finally, as $f(x)$ is improperly integrable on $(0,\,\infty)$ so is $p\;f(x)$ and consequently $f(x)\,g(x) = f(x)\,(g(x)-p) + p\,f(x)$ is improperly integrable on $(0,\infty)$. $\square$

Exercise 3. Formulate and prove Theorem 2 and Theorem 3 for the general interval $(a,\,b)$ instead of $(0,\infty)$.

Example 4. Prove that $\frac{\sin x}{1+x^{1/2}}$ is improperly integrable on $(0,\infty)$.

Proof. We take $f(x)=\sin x$ and $g(x)=\frac{1}{1+x^{1/2}}$. Clearly the conditions of Theorem 1 is satisfied. $\square$

- Calculation of $\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x$.

Set

$$g(y) := \int_0^\infty e^{-xy}\,\frac{\sin x}{x}\,\mathrm{d}x. \tag{5}$$

Exercise 4. Prove that $g(y)$ is defined for all $y>0$.

We calculate

$$g'(y) = \int_0^\infty (-x)\,e^{-xy}\frac{\sin x}{x}\,\mathrm{d}x = -\int_0^\infty e^{-xy}\sin x\,\mathrm{d}x = -\frac{1}{1+y^2}. \tag{6}$$

Problem 1. Justify (6).

This gives

$$g(y) = C - \arctan y. \tag{7}$$

Problem 2. Prove

$$\lim_{y\to 0+} g(y) = \int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x.$$

From (7) we have $\lim_{y\to 0+} g(y) = C$. Therefore all we need is the value of C.

Exercise 5. Prove that

$$\lim_{y\to\infty} \int_0^\infty e^{-xy} \frac{\sin x}{x}\, \mathrm{d}x = 0. \tag{8}$$

Taking $y \to \infty$ in (7) we see that $C = \frac{\pi}{2}$. Consequently $\displaystyle\int_0^\infty \frac{\sin x}{x}\, \mathrm{d}x = \frac{\pi}{2}$.

- Relation to Infinite Series.

  Consider the "harmonic numbers" $H_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. From

$$\frac{1}{2} = \int_1^2 \frac{\mathrm{d}x}{2} < \int_1^2 \frac{\mathrm{d}x}{x} < \int_1^2 \frac{\mathrm{d}x}{1} = 1 \tag{9}$$

$$\frac{1}{3} = \int_2^3 \frac{\mathrm{d}x}{3} < \int_2^3 \frac{\mathrm{d}x}{x} < \int_2^3 \frac{\mathrm{d}x}{2} = \frac{1}{2} \tag{10}$$

$$\vdots \qquad \vdots \qquad \vdots$$

we see that

$$H_n - 1 < \int_1^n \frac{\mathrm{d}x}{x} < H_{n-1} < H_n. \tag{11}$$

Exercise 6. Use (11) to prove the divergence of $\sum_{n=1}^\infty \frac{1}{n}$.

Exercise 7. Use similar idea to study the convergence/divergence of $\displaystyle\sum_{n=2}^\infty \frac{1}{n\,(\ln n)^p}$, $p \in \mathbb{R}$.

Rearranging the terms we have

$$\gamma_n := H_n - \ln n \in (0, 1). \tag{12}$$

As

$$\gamma_{n+1} - \gamma_n = \int_n^{n+1} \left[\frac{1}{n+1} - \frac{1}{x}\right] \mathrm{d}x < 0, \tag{13}$$

γ_n is decreasing and therefore there is $\gamma \in (0, 1)$ such that $\lim_{n\to\infty} \gamma_n = \gamma$. This number is called the "Euler-Mascheroni" constant, about which very little is known. We don't even know whether it is rational or not.