Math 118 Winter 2015 Homework 5

Due Thursday Feb. 26 3pm in Assignment Box

Question 1. (5 pts) *Prove the following* ***by definition***.

a) (2 pts) $\dfrac{1}{1+x^2}$ *is improperly integrable on* $(0,\infty)$.

b) (3 pts) $\tan x$ *is not improperly integrable on* $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

Question 2. (5 pts) *Let* $|f|$ *be improperly integrable on* (a,b) *and* g *be locally integrable on* (a,b). *Further assume that* g *is bounded on* (a,b). *Prove that* fg *is improperly integrable on* (a,b).

Question 3. (5 pts) *Let* $f(x):[1,\infty)\mapsto\mathbb{R}$ *be positive and decreasing. Denote* $a_n:=f(n)$ *for* $n\in\mathbb{N}$. *Prove*

$$\sum_{n=1}^{\infty} a_n \text{ converges} \Longleftrightarrow f(x) \text{ is improperly integrable on } (1,\infty). \tag{1}$$

Question 4. (5 pts + **5 pts**) *Consider the function*

$$g(y):=\int_0^\infty e^{-xy}\,\frac{\sin x}{x}\,\mathrm{d}x. \tag{2}$$

a) (3 pts) *Prove that* $g(y)$ *is defined for all* $y>0$. *(Hint: Prove* $\left|\frac{\sin x}{x}\right|\leqslant 1$*)*

b) (2 pts) *Prove that* $\lim_{y\to\infty} g(y)=0$.

c) (**2 extra pts**) *Prove*

$$\lim_{y\to 0+} g(y)=\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x. \tag{3}$$

d) (**3 extra pts**) *Prove that* $g(y)$ *is differentiable on* $(0,\infty)$ *with* $g'(y)=-\frac{1}{1+y^2}$.

Note. *You should prove directly and not use theorems from multi-variable calculus.*