Math 118 Winter 2015 Lecture 22 (Feb. 12, 2015)

- Properties of Improper Integrals.
  - FTC.

THEOREM 1. *Let f be improperly integrable on (a, b) and let $F' = f$ on (a, b). Then $F(b-) := \lim_{d \to b-} F(d)$ and $F(a+) := \lim_{c \to a+} F(c)$ exist and furthermore*

$$\int_a^b f(x)\,\mathrm{d}x = F(b-) - F(a+). \tag{1}$$

Exercise 1. Prove Theorem 1.

THEOREM 2. *Let f be improperly integrable on (a, b). Define $G: (a, b) \mapsto \mathbb{R}$ by $G(x) := \int_a^x f(t)\,\mathrm{d}t$. Then*

a) $G(x)$ is continuous on (a, b);

b) if furthermore $f(x)$ is continuous at $x_0 \in (a, b)$, then $G(x)$ is differentiable at x_0 with $G'(x_0) = f(x_0)$.

Exercise 2. Prove Theorem 2.

- Integration by parts.

THEOREM 3. *Let $u(x)$, $v(x)$ be such that $u\,v'$ and $u'\,v$ are improperly integrable on (a, b). Then*

$$\lim_{d \to b-} u(d)\,v(d) \ \ and \ \ \lim_{c \to a+} u(c)\,v(c) \tag{2}$$

exist and

$$\int_a^b u\,v'\,\mathrm{d}x = \lim_{d \to b-} u(d)\,v(d) - \lim_{c \to a+} u(c)\,v(c) - \int_a^b u'\,v\,\mathrm{d}x. \tag{3}$$

Exercise 3. Prove Theorem 3.

Exercise 4. Let $|f(x)|$ be improperly integrable on (a, b) and $g(x)$ be locally integrable and bounded on (a, b). Prove that $f(x)\,g(x)$ is improperly integrable on (a, b).

Exercise 5. Justify integration by parts for

$$\int_0^\infty e^{-x} \cos x\,\mathrm{d}x, \qquad \int_0^\infty x^{100}\,e^{-x}, \qquad \int_0^1 (x^2 + 1) \ln x\,\mathrm{d}x. \tag{4}$$

- Change of variables.

THEOREM 4. *Let $u(x)$ be differentiable and monotone on (a, b) with $u'(x)$ continuous on (a, b). Let $f(x)$ be continuous and improperly integrable on $u((a, b))$. Further assume that $f(u(t))\,u'(t)$ is improperly integrable on (a, b). Then there holds*

$$\int_a^b f(u(t))\,u'(t)\,\mathrm{d}t = \int_{u(a)}^{u(b)} f(x)\,\mathrm{d}x. \tag{5}$$

Exercise 6. Prove Theorem 4.

- Proving convergence.
  - Dominance

THEOREM 5. *Let $f\colon (a,b) \mapsto \mathbb{R}$. Assume*

i. *f is locally integrable;*
ii. *There is $g\colon (a,b) \mapsto \mathbb{R}$ such that $|f(x)| \leqslant g(x)$ for all $x \in (a,b)$;*
iii. *$g(x)$ is improperly integrable on (a,b).*

Then $f(x)$ is improperly integrable on (a,b).

Remark 6. Note that in particular the improper integrability of $|f|$ implies the improper integrability of f.

Proof. Let $[c,d] \subset (a,b)$ be arbitrary. As $g(x)$ is improperly integrable on (a,b), the limit

$$\lim_{d \to b-} \int_c^d g(x)\,\mathrm{d}x \tag{6}$$

exists and $G(d) := \int_c^d g(x)\,\mathrm{d}x$ is Cauchy. Since

$$\left| \int_c^{d_1} f(x)\,\mathrm{d}x - \int_c^{d_2} f(x)\,\mathrm{d}x \right| \leqslant \int_{d_1}^{d_2} g(x)\,\mathrm{d}x \tag{7}$$

we see that $\int_c^d f(x)\,\mathrm{d}x$, as a function of d, is Cauchy. Therefore $\lim_{d \to b-} \int_c^d f(x)\,\mathrm{d}x$ exists. Similarly we can prove $\lim_{c \to a+} \left[\lim_{d \to b-} \int_c^d f(x)\,\mathrm{d}x \right]$ exists and the conclusion follows. □

COROLLARY 7. *Let $f, g\colon (a,b) \mapsto \mathbb{R}$. Assume*

a) *$f(x) \geqslant g(x) \geqslant 0$ for all $x \in (a,b)$;*
b) *$g(x)$ is not improperly integrable on (a,b).*

Then $f(x)$ is not improperly integrable on (a,b).

Exercise 7. Prove Corollary 7.

Example 8. Study the improper integrability of $f(x) := \frac{1}{1+x^\alpha}$, $\alpha \in \mathbb{R}$ on $(0,\infty)$.

Solution.

– $\alpha \leqslant 1$. Let

$$g(x) := \begin{cases} 0 & x \leqslant 1 \\ \dfrac{1}{2\,x} & x > 1 \end{cases}. \tag{8}$$

Exercise 8. Prove that $g(x)$ is not improperly integrable on $(0,\infty)$.

As $f(x) \geqslant g(x) \geqslant 0$ for all $x \in (0,\infty)$, we see that $f(x)$ is not improperly integrable on $(0,\infty)$.

– $\alpha > 1$. Let

$$g(x) := \begin{cases} 1 & x \leqslant 1 \\ \dfrac{1}{x^\alpha} & x > 1 \end{cases}. \tag{9}$$

Then $|f(x)| \leqslant g(x)$ for all $x \in (0,\infty)$ and f is thus improperly integrable on $(0,\infty)$.

Exercise 9. Prove that if $\lim_{x \to \infty} \frac{|f|}{g} = l \in \mathbb{R}$, $g \geqslant 0$ and improperly integrable on $(0,\infty)$, then f is improperly integrable on $(0,\infty)$.