MATH 118 WINTER 2015 LECTURE 21 (FEB. 11, 2015)

- Recall: Improper (Riemann) Integrals.
  - (LOCAL INTEGRABILITY) A function $f: (a, b) \mapsto \mathbb{R}$ is said to be "locally integrable" on (a, b) if and only if for every $[c, d] \subset (a, b)$, f is Riemann integrable on $[c, d]$.
  - (IMPROPER INTEGRABILITY) A function f: $(a, b) \mapsto \mathbb{R}$ is said to be "improperly integrable" on (a, b) if and only if
    i. f is locally integrable on (a, b);
    ii. $A := \lim_{d \to b-} \left[\lim_{c \to a+} \int_c^d f(x)\, \mathrm{d}x \right]$ (or $\lim_{c \to a+} \left[\lim_{d \to b-} \int_c^d f(x)\, \mathrm{d}x \right]$) exists and is finite.

    We denote $\int_a^b f(x)\, \mathrm{d}x = A$.
  - (SIMPLIFIED INTEGRABILITY CHECK)
    - If f is Riemann integrable on (a, b) then it is improperly integrable on (a, b).
    - If f is Riemann integrable on $[a, d]$ for every $d \in (a, b)$, then f is improperly integrable on (a, b) if and only if $A := \lim_{d \to b-} \int_a^d f(x)\, \mathrm{d}x$ exists and is finite.
    - If f is Riemann integrable on $[c, b]$ for every $c \in (a, b)$, then then f is improperly integrable on (a, b) if and only if $A := \lim_{c \to a+} \int_c^b f(x)\, \mathrm{d}x$ exists and is finite.
- Properties.
  - Integrability.

    Exercise 1. Find a continuous $f: (a, b) \mapsto \mathbb{R}$ such that f is not improperly integrable on (a, b). What if we further require (a, b) to be bounded? (Hint:[1])

    Exercise 2. Find a decreasing function f: $(0, \infty)$ such that $\lim_{x \to \infty} f(x) = 0$ but f is not improperly integrable on (a, b). (Hint:[2])
  - Arithmetics.

    Exercise 3. Let $f, g: (a, b) \mapsto \mathbb{R}$ be improperly integrable. Let $r, s \in \mathbb{R}$ be arbitrary. Prove that $r\, f + s\, g$ is improperly integrable on (a, b) and furthermore $\int_a^b [r\, f + s\, g]\, \mathrm{d}x = r \int_a^b f(x)\, \mathrm{d}x + s \int_a^b g(x)\, \mathrm{d}x$.

    Example 1. Let $f(x) := \frac{\sin x}{x}$. Then $f(x)$ is improperly integrable on (π, ∞) but $|f(x)|$ is not.

    Proof. It is clear that $\frac{\sin x}{x}$ is integrable on $[\pi, d]$ for every $d < \infty$. Now we integrate by parts

$$\begin{aligned} \int_\pi^d \frac{\sin x}{x}\, \mathrm{d}x &= \frac{-\cos x}{x}\Big|_\pi^d - \int_\pi^d \frac{\cos x}{x^2}\, \mathrm{d}x \\ &= -\frac{\cos d}{d} - \frac{1}{\pi} - \int_\pi^d \frac{\cos x}{x^2}\, \mathrm{d}x. \end{aligned} \tag{1}$$

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1. $\sin x$.
2. $1/x$.

As $\lim_{d\to\infty}\frac{1}{\pi}=\frac{1}{\pi}$ and

$$-\frac{1}{d}\leqslant\frac{\cos d}{d}\leqslant\frac{1}{d}\xrightarrow{\text{Squeeze}}\lim_{d\to\infty}\frac{\cos d}{d}=0, \tag{2}$$

all we need to prove now is the existence of $\lim_{d\to\infty}F(d)$ where $F(d):=\int_{\pi}^{d}\frac{\cos x}{x^2}\,\mathrm{d}x$. We show that $F(d)$ is Cauchy.

Let $\varepsilon>0$ be arbitrary. Take $d_0:=\varepsilon^{-1}$. Then for every $d_2>d_1>d_0$ we have

$$\begin{aligned}|F(d_2)-F(d_1)| &= \left|\int_{d_1}^{d_2}\frac{\cos x}{x^2}\,\mathrm{d}x\right|\\ &\leqslant \int_{d_1}^{d_2}\frac{|\cos x|}{x^2}\,\mathrm{d}x\\ &\leqslant \int_{d_1}^{d_2}\frac{\mathrm{d}x}{x^2}\\ &\leqslant \frac{1}{d_1}\\ &< \frac{1}{d_0}=\varepsilon.\end{aligned} \tag{3}$$

Therefore $F(d)$ is Cauchy and the proof ends.

Now we show that $\left|\frac{\sin x}{x}\right|$ is not improperly integrable on (π,∞). All we need to show is that $G(d):=\int_{\pi}^{d}\left|\frac{\sin x}{x}\right|\mathrm{d}x$ does not have a finite limit as $d\longrightarrow\infty$. Take $d_n:=n\,\pi$. We have

$$\begin{aligned}G(d_n) &= \int_{\pi}^{n\pi}\frac{|\sin x|}{x}\,\mathrm{d}x\\ &= \sum_{k=1}^{n-1}\int_{k\pi}^{(k+1)\pi}\frac{|\sin x|}{x}\,\mathrm{d}x\\ &\geqslant \sum_{k=1}^{n-1}\int_{k\pi}^{(k+1)\pi}\frac{|\sin x|}{(k+1)\,\pi}\,\mathrm{d}x\\ &\overset{t=x-k\pi}{=\!=} \sum_{k=1}^{n-1}\frac{1}{(k+1)\,\pi}\int_{0}^{\pi}|\sin x|\,\mathrm{d}x\\ &= \frac{2}{\pi}\sum_{k=1}^{n-1}\frac{1}{k+1}.\end{aligned} \tag{4}$$

Thus $\lim_{n\to\infty}G(d_n)=\infty$ and the conclusion follows. □

Exercise 4. Prove that $f(x)=\frac{\sin x}{x}$ is improperly integrable on $(0,\infty)$.

Remark 2. The above example shows that, in contrast to Riemann integrability, the improper integrability of f does not imply the improper integrability of $|f|$. We will see later that the improper integrability of f, g does not imply the improper integrability of the product $f\,g$ either.