MATH 118 WINTER 2015 LECTURE 20 (FEB. 9, 2015)

Note. In the following $a, b \in \mathbb{R} \cup \{\pm\infty\}$.

- Improper (Riemann) Integrals

DEFINITION 1. (LOCAL INTEGRABILITY) *A function $f: (a, b) \mapsto \mathbb{R}$ is said to be "locally integrable" on (a, b) if and only if for every $[c, d] \subset (a, b)$, f is Riemann integrable on $[c, d]$.*

Exercise 1. Find a function that is locally integrable but not Riemann integrable on $(0, 1)$.

Exercise 2. Prove that every $[c, d] \subset (a, b)$ must be finite, even though a (or b or both) could be infinity.

Remark 2. The reason why the above is called "local integrability" is that it is equivalent to the following requirement:

> For every $u \in (a, b)$, there is $\delta > 0$ such that f is Riemann integrable on $[u - \delta, u + \delta]$.

Problem 1. Prove that f is locally integrable on (a, b) if and only if f meets the above requirement. (Hint:[1])

DEFINITION 3. (IMPROPER INTEGRABILITY) *A function $f: (a, b) \mapsto \mathbb{R}$ is said to be "improperly integrable" on (a, b) if and only if*

i. f is locally integrable on (a, b);

ii. $A := \lim_{d \to b-} \left[\lim_{c \to a+} \int_c^d f(x)\,\mathrm{d}x \right]$ exists and is finite.

We denote $\int_a^b f(x)\,\mathrm{d}x = A$.

- Examples.

Example 4. Prove by definition that e^{-x} is improperly integrable on $(0, \infty)$.

Proof. First let $[c, d] \subset (0, \infty)$ be arbitrary. Then $c > 0$ and $d < \infty$ so $[c, d]$ is bounded. As e^{-x} is continuous on the bounded closed interval $[c, d]$ it is integrable on $[c, d]$. So e^{-x} is locally integrable on $(0, \infty)$.

Next we calculate

$$\lim_{d \to \infty} \left[\lim_{c \to 0+} \int_c^d e^{-x}\,\mathrm{d}x \right] = \lim_{d \to \infty} \left[\lim_{c \to 0+} [e^{-c} - e^{-d}] \right] = \lim_{d \to \infty} [1 - e^{-d}] = 1. \tag{1}$$

Therefore e^{-x} is improperly integrable on $(0, \infty)$ with $\int_0^\infty e^{-x}\,\mathrm{d}x = 1$. □

Example 5. Prove by definition that $\ln x$ is improperly integrable on $(0, 1)$.

Proof. First let $[c, d] \subset (0, 1)$ be arbitrary. Then $c > 0, d < 1$ and $[c, d]$ is bounded. As $\ln x$ is continuous on $[c, d]$, it is integrable on $[c, d]$ and thus locally integrable on $(0, 1)$.

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1. First show that there is a sequence $\{u_n\}$ such that $[c, d] \subset \cup_{n \in \mathbb{N}} (u_n - \delta_n, u_n + \delta_n)$. Then use Bolzano-Weierstrass to prove that $[c, d]$ is contained in a finite union of some $(u_n - \delta_n, u_n + \delta_n)$.

Next we calculate

$$\lim_{d \to 1-} \left[\lim_{c \to 0+} \int_c^d \ln x \, \mathrm{d}x \right] = \lim_{d \to 1-} \left[\lim_{c \to 0+} [d \ln d - c \ln c - d + c] \right] = \lim_{d \to 1-} [d \ln d - d] = -1. \tag{2}$$

So $\ln x$ is improperly integrable on $(0, 1)$ with $\int_0^1 \ln x \, \mathrm{d}x = -1$. □

Exercise 3. Why does $\lim_{c \to 0+} c \ln c = 0$?

Example 6. Prove by definition that $\sin x$ is not improperly integrable on $(0, \infty)$.

Proof. We calculate

$$\lim_{d \to \infty} \left[\lim_{c \to 0+} \int_c^d \sin x \, \mathrm{d}x \right] = \lim_{d \to \infty} \left[\lim_{c \to 0+} [\cos c - \cos d] \right] = \lim_{d \to \infty} [1 - \cos d]. \tag{3}$$

As $\lim_{d \to \infty} [1 - \cos d]$ does not exist, $\sin x$ is not improperly integrable on $(0, \infty)$. □

- Theoretical issues.
  - Improper integrals generalize Riemann integrals.

    PROPOSITION 7. *Let $f: [a, b] \mapsto \mathbb{R}$ be Riemann integrable on $[a, b]$ with $\int_a^b f(x) \, \mathrm{d}x = A \in \mathbb{R}$. Then it is improperly integrable on (a, b) with improper integral A.*

    Exercise 4. Prove Proposition 7. (Hint:[2])

  - The order of the limits does not matter.

    PROPOSITION 8. *Let f be locally integrable on (a, b). Then*

$$\lim_{d \to b-} \left[\lim_{c \to a+} \int_c^d f(x) \, \mathrm{d}x \right] = A \Longleftrightarrow \lim_{c \to a+} \left[\lim_{d \to b-} \int_c^d f(x) \, \mathrm{d}x \right] = A. \tag{4}$$

    Exercise 5. Prove Proposition 8. (Hint:[3])

  - Sometimes only one limit is needed.

    PROPOSITION 9. *Let f be integrable on $[a, d]$ for every $d \in (a, b)$. Then f is improperly integrable on (a, b) if and only if $A := \lim_{d \to b-} \int_a^d f(x) \, \mathrm{d}x$ exists and is finite. Furthermore the improper integral $\int_a^b f(x) \, \mathrm{d}x = A$. A similar result holds when f is integrable on $[c, b]$ for every $c \in (a, b)$.*

    Exercise 6. Prove Proposition 9.

2. FTC2. Don't forget to prove local integrability first.

3. Take an arbitrary $e \in (a, b)$. Fix it. Write $\int_c^d f(x) \, \mathrm{d}x = \int_c^e f(x) \, \mathrm{d}x + \int_e^d f(x) \, \mathrm{d}x$.